TCP Networks Stabilized by Buffer-Based AQMs

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Abstract— **In this work we develop stability conditions for congestion control of the present Internet characterized by TCP-controlled sources and buffer-based active queue management (AQM) schemes. Prevailing stability results are geared towards rate-based AQMs and are not applicable to the class of networks considered here. Our new conditions can be expressed entirely in terms of network parameters, i.e., routing, which make them useful in the design of buffer-based AQMs such as RED, REM and PI.**

Key Words: Control theory.

I. Introduction

Recent empirical studies in [1] demonstrate the efficacy of Active Queue Management (AQM), in conjunction with Explicit Congestion Notification (ECN), to significantly improve Internet performance by reducing response time and by allowing provider-links to operate near saturation without degradation in user-perceived performance. Our present work is concerned with a network-wide stability analysis of such AQM networks, focusing on those using buffer lengths as a measure of congestion.

Currently, the most-widely proposed alternative to drop-tail routers is Random Early Detection (RED) algorithm [2] which computes local drop or mark probabilities based on local average buffer length. As a feedback mechanism, TCP-controlled sources[1] react to these drops or marks by adapting their window sizes. Thus, to analyze congestion control of the current Internet (as depicted in Figure 1), one should consider the feedback interaction between buffer-based AQMs and TCP source algorithms. To date, there are no results specific to this situation, and this serves as motivation for our present work. Before giving an overview of related results, we first discuss the existing analysis results for congestion control.

[1]Throughout this paper the term TCP-controlled sources refers to AIMD like sources (e.g. TCP-Reno, TCP-SACK).

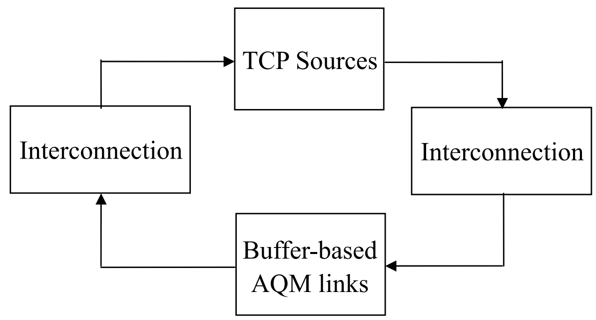

Fig. 1. TCP/AQM Diagram

To begin, we distinguish between rate-based and buffer-based AQMs. Rate-based AQMs use aggregate incoming flow rates to communicate congestion with a goal of controlling these rates to some fraction of link capacity. In achieving this, buffers become empty, queuing delays become nonexistent and round trip times (RTTs) become fixed by propagation delay. The Adaptive Virtual Queue (AVQ) [3] is the prototypical rate-based AQM scheme. In contrast, when buffer lengths are used to communicate congestion, both buffer dynamics and queuing delays come into play. A number of AQMs are buffer-based, such as the static law in [4] and dynamic schemes including RED [2], Random Early Marking (REM) [5] and proportional-integral (PI) [4]. From a control-theoretic view point, RED amounts to gain and low-pass filtering, while AVQ, REM and PI laws incorporate elements of gain, integration and phase lead.

Stability analysis of congested networks has been primarily conducted for rate-based AQMs. For example, a globally-stable, static, rate-based AQM was proposed in [6] for a delay-free network. This was extended in [7] to give a local stability condition when all RTTs were equal, and, in [9], generalized to handle heterogeneous RTTs. In [10]-[11], these were further generalized to allow for averaging AQMs. In [12], a stability condition was derived for AVQ while in [13], an integral rate-based AQM was considered. The use of rate-based AQM, and TCP-like source dynamic, common to the aforementioned research, leads to an overall network dynamic comprised of two decentralized dynamical systems, one each for the TCP-controlled sources and the AQM links, both dynamics interconnected by routing. As we shall see in Section 2, this two-dynamic structure is not rich enough to describe buffer-based congestion-control systems that involve three decentralized dynamics.

The inadequacy of existing rate-based stability proofs prompts the need for new arguments to establish stability in today's Internet which relies on buffer-based AQMs. While stability analysis for RED and PI AQMs have been made in [4] for a single congested link, they are yet to be generalized to the network case. In this paper, we present a new, local stability condition for buffer-based TCP/AQM networks with arbitrary topologies and heterogeneous delays. These conditions display the interplay between TCP and AQM gain, and network delay. They can be recast as design rules for RED and PI AQMs, couched solely in terms of network parameters including a so-called *gain of routing*. The conditions can be readily extended to allow for rate-based AQMs and reduce to the expected relations for a single link network [4]. This defines the contribution of our work.

The remainder of the paper is organized as follows. In Section 2, we state our stability condition for buffer-based TCP/AQM networks. In Section 3, we apply these condition to the design of RED and PI AQM's. In Section 4, we discuss the implication of the routing gain on congestion control and illustrate these effects via `ns` simulations. A number of appendices include details of theoretical arguments supporting our results.

Notation: For vector $x \in \mathbf{C^n}$, $x_k \in \mathbf{C}$ denotes its kth component. $X = diag\{x\}$ is a diagonal matrix in $\mathbf{C^{n\times n}}$ with the x_k's as diagonal elements. For set $S \subset \mathbf{C}$, $Co(S)$ denotes its convex hull. Addition, multiplication and division of complex sets is taken in the usual sense. $\mathcal{F}(X)$ is the field of values of $X \in \mathbf{C^{n\times n}}$; i.e., $\mathcal{F}(X) = \{x^H X x, x \in \mathbf{C}^{n\times 1}, x^H x = 1\}$. Also, $\lambda(X)$, $\rho(X)$, $\sigma_{min}[X]$ and $\sigma_{max}[X]$ denote the eigenvalues, spectral radius, and, smallest and largest singular values of X respectively. Finally, $disc(r)$ denotes the closed disk $\{s \in \mathbf{C} : |s| \le r\}$.

II. Stability Condition for Buffer-Based AQM Networks

Consider a network consisting of m congested links and n heterogeneous source sessions. The interconnections are described by the matrix $\mathcal{R}(s) = [R_{ik}(s)]$ where

$$\mathcal{R}_{ik}(s) = \begin{cases} e^{-s\overrightarrow{\tau_{ik}}} & \text{if source } k \text{ traverses link } i \\ 0 & \text{otherwise} \end{cases}$$

and where $\mathcal{R}(0)$ is full rank. Here $\overrightarrow{\tau_{ik}}$ denotes the forward delay from source k to link i. The so-called *routing matrix* is $\mathcal{R}(0)$. Letting $\overleftarrow{\tau_{ik}}$ denote the return delay from link i to source k, we define the *round-trip time* of source k by $\tau_k = \overrightarrow{\tau_{ik}} + \overleftarrow{\tau_{ik}}$.

Using the TCP fluid model [8], the congestion window for the kth source is approximated by the nonlinear differential equation

$$\dot{w}_k(t) = \frac{1}{\tau_k} - \frac{w_k^2(t)}{2\tau_k} \sum_{i=1}^{m} \left[\mathcal{R}(0)^T\right]_{ki} p_i(t - \overleftarrow{\tau_{ik}}) \quad (1)$$

where $w_k(t)$ and $p_i(t)$ are the congestion window size of the kth source and the packet marking probability at the ith link, respectively.[2] We model the ith congested link by

$$\dot{q}_i(t) = -c_i I_{q>0} + \sum_{k=1}^{n} [\mathcal{R}_{ik}(0)] \frac{\eta_k w_k(t - \overrightarrow{\tau_{ik}})}{\tau_k} \quad (2)$$

where c_i is the ith link's capacity, η_k the number of sessions in the kth source, and $I_{q>0}$ the indicator function. Linearizing (1)-(2) about equilibrium $(\hat{w}, \hat{q}, \hat{p})$ and taking Laplace transforms results in the source and link dynamics:

$$w(s) = -\tilde{F}(s)\mathcal{R}(-s)^T p(s) \quad (3)$$

and

$$q(s) = (sI + \Omega)^{-1}\mathcal{R}(s) N \hat{T}^{-1} w(s) \quad (4)$$

where $\tilde{F}(s) = diag\{\tilde{f}_k(s)\}$,

$$\tilde{f}_k(s) = \frac{e^{-s\tau_k}}{s + \frac{2}{\hat{w}_k \tau_k}} \frac{1}{\tau_k \hat{u}_k},$$

$\hat{u} = \mathcal{R}(0)\hat{p}$ is equilibrium aggregate marking probability, and where

$$\Omega = \mathcal{R}(0) N \hat{W} \hat{T}^{-2} \mathcal{R}(0)^T C^{-1},$$

[2]The original fluid-flow models in [8] contain the delayed term $w_k(t - \tau_k)$ in the right-hand side of (1). As in [4], we ignore this delay which appears to have second-order affect, especially when equilibrium congestion windows are large.

$\hat{T} = diag\{\tau\}, N = diag\{\eta\}, \hat{W} = diag\{\hat{w}\}$ and $C = diag\{c\}$. Next, we let $k_i(s)$ denote the linearized AQM dynamics at link i so that

$$p(s) = K(s)q(s)$$

where $K(s) = diag\{k(s)\}$. We scale K by C to obtain

$$D(s) = K(s)C \tag{5}$$

where $D(s) = diag\{d(s)\}$. We would like to emphasize here that each link is allowed to select its AQM parameters independent from all other AQMs in the network.

Finally, to invoke the Generalized Nyquist Stability Criterion [14] we first form the following return ratio function for system (3) – (5):

$$\begin{aligned} L(s) &= \tilde{F}(s)\mathcal{R}(-s)^T C^{-\frac{1}{2}} D(s) \\ &\quad \cdot (VG(s)V^H)^{-1} C^{-\frac{1}{2}} \mathcal{R}(s) N \hat{T}^{-1} \end{aligned} \tag{6}$$

where $G(s) = diag\{g(s)\}$, $g_i(s) = s + \lambda_i$, and $\lambda_i > 0$ is an eigenvalue of $C^{-\frac{1}{2}}\Omega C^{\frac{1}{2}}$; see Appendix A for details.

Remark: Compared with rate-based TCP/AQM networks that have two dynamics, $F(s)$ for source dynamics and $D(s)$ for AQM dynamics, the network (6) has an additional term, $(VG(s)V^H)^{-1}$, which arises from queuing dynamics. Specifically, source, AQM and queuing dynamics are coupled through $\mathcal{R}$ and V in (6). The results in [11]-[13] and the arguments underlying the related proofs can not be modified to be made applicable to this setup. This characterizes the unique aspects of the problem considered here.

We now present our main result on local stability of the network described in (3)–(5). To state this result we introduce some additional notation. First, we take $F(j\omega) = diag\{f(j\omega)\}$ where $f_k(j\omega) = \frac{M_k}{\hat{W}_k}\tilde{f}_k(j\omega)$ and then let $\sigma_{min}[\tilde{\mathcal{R}}]$ denote the minimal singular value of

$$\tilde{\mathcal{R}} = C^{-\frac{1}{2}}\mathcal{R}(j\omega)(N\hat{W}M^{-1}\hat{T}^{-1})^{\frac{1}{2}}$$

where $M = diag\{\mu\}$ where μ_k is the number of congested links traversed by the kth source.

Theorem 1. *Consider the linearized network consisting of n sources and m links described in $(3)-(5)$. Assume the routing matrix $\mathcal{R}(0)$ has full rank. Then, the network is locally stable if for each $\omega \geq 0$, there exists a $d_\omega \in \boldsymbol{C}$ such that the set*

$$\begin{aligned} S(\omega) &= \frac{Co\{\sigma^2_{min}[\tilde{\mathcal{R}}], 1\}Co\{d_\omega f_k\}}{Co\{g_i\}} \\ &\quad + \frac{disc(\|F\|_2 \, \|D - d_\omega I\|_2)}{Co\{g_i\}} \end{aligned} \tag{7}$$

does not intersect $(-\infty, -1]$.

Proof of Theorem 1: Since $L(s)$ is stable, it suffices to invoke the Generalized Nyquist Theorem [14] and show that the eigenvalues of $L(j\omega)$ do not intersect $(-\infty, -1]$ for all $\omega \geq 0$. The proof will show that these eigenvalues lie in $S(\omega)$. Because eigenvalues are invariant under matrix commutation,

$$\lambda(L(j\omega)) = \lambda\left(\tilde{\mathcal{R}}F(j\omega)\tilde{\mathcal{R}}^H D(j\omega)(VG(j\omega)V^H)^{-1}\right).$$

Since $\mathcal{F}(V^H G(j\omega)V) = \mathcal{F}(G(j\omega)) = Co\{g\}$ (recall that $R(0)$ is full rank), then $0 \notin \mathcal{F}(V^H G(j\omega)V)$. Hence, from [15]

$$\begin{aligned} \lambda\left(\tilde{\mathcal{R}}F\tilde{\mathcal{R}}^H D(VGV^H)^{-1}\right) &\subset \\ &\mathcal{F}(\tilde{\mathcal{R}}F(s)\tilde{\mathcal{R}}^H D(s))/\mathcal{F}(V^H G(s)V). \end{aligned}$$

From Appendix B

$$\begin{aligned} \mathcal{F}(\tilde{\mathcal{R}}F\tilde{\mathcal{R}}^H D) &\subseteq \\ &Co\{\sigma^2_{min}[\tilde{\mathcal{R}}], 1\}Co\{df_j\} + disc(\|F\|_2\|D - dI\|_2). \end{aligned}$$

Since $\mathcal{F}(V^H G(s)V) = Co\{g_i\}$, then $\lambda(L(j\omega)) \subset S(\omega)$ and the proof is completed. □

Remark: We observe an interplay between the dynamics of sources, queuing and AQMs in (7). Specifically, if the gain of sources' dynamics $\|F\|$ becomes larger and/or the links' poles λ_i becomes smaller, then the maximal gain of the AQMs $|d_i(j\omega)|$ must be decreased commensurably. The scaler d_ω may be used to achieve larger, if possible, AQM gains without sacrificing stability.

We now specialize our stability condition to i) a single congested link, and ii) a rate-based AQM.

A. Stability condition for network with single congested link

For one congested link with one TCP source, we have $m = 1$, $\sigma^2_{min}[\tilde{\mathcal{R}}] = 1$ and $M = 1$. For the stability condition in Theorem 1 we select the free parameter d_ω to be the frequency response of the single link AQM; i.e., $d_\omega \equiv d(j\omega)$. Then, (7) becomes

$$\begin{aligned} S(\omega) &= \frac{d(j\omega)f(j\omega)}{g(j\omega)} \\ &= \frac{d(j\omega)\frac{C}{2N}e^{-j\omega\tau}}{(j\omega + \frac{2N}{\tau^2 C})(j\omega + \frac{1}{\tau})}. \end{aligned}$$

This is precisely the stability condition obtained in [4].

B. Stability condition for rate-based TCP/AQM networks

For rate-based TCP/AQM networks, it is easy to show $G(s) = I$. The return ratio function (6) then becomes

$$L_{rate}(j\omega) = \tilde{\mathcal{R}} F(j\omega) \tilde{\mathcal{R}}^H D(j\omega). \tag{8}$$

Corollary 1 (See Appendix C for proof): *If* $0 \notin Co\left(d_k^{-1}(j\omega)\right)$ *and*

$$S_{rate}(\omega) = \frac{Co\{\sigma_{min}^2[\tilde{\mathcal{R}}], 1\} Co\{f_j\}}{Co\left(d_k^{-1}(j\omega)\right)} \tag{9}$$

does not intersect $(-\infty, -1]$ *for all* $\omega \geq 0$*, then the rate-based network associated with* L_{rate} *is locally stable.* □

Remark: Note that (9) is comparable with Lemma 1 in [11] which relies on the condition $\rho(|R|^T|R|) < 1$ (R in [11] is $\tilde{\mathcal{R}}$ here). This condition can be relaxed to a condition requiring the diagonal elements of RR^H to be bounded by 1. With this relaxation, applying Lemma 1 in [11] to the system (8) amounts to replacing the set in (9) with

$$Co\{(Co\{\pm\sqrt{f_i d_k} : R_{ik} \neq 0\})^2\}. \tag{10}$$

We observe some similarity between $\frac{Co\{f_j\}}{Co\left(d_k^{-1}(j\omega)\right)}$ and $Co\{(Co\{\pm\sqrt{f_i d_k} : R_{ik} \neq 0\})^2\}$, but a precise comparison is yet to be developed.

III. Design Rules for RED & PI AQMs

In this section we show that Theorem 1 can be used to provide design rules for tuning RED and PI AQMs. The rules will be expressed in terms of network parameters.

A. Design stabilizing RED

A RED AQM computes the packet-marking probability as a function of average queue length, and, as in [4], can be modeled as the low-pass filter

$$d_i(s) = \frac{l_i}{s + p_i} \tag{11}$$

where l_i is the AQM gain and $1/p_i$ is the low-pass filter time constant. We now show how to select these parameters for stability.

Proposition 1 (See Appendix D for proof): *Consider an* m*-link,* n*-source TCP/RED network described by* (3) − (5) *and* (11)*. Let the RED parameters be chosen as:* $l_i \in (0,\ \ell_{max})$ *and* $p_i \in [P_{\min},\ P_{\max}]$*. Then, this network is locally stable if*

$$\ell_{max} \leq 2\frac{\min\{N_k \hat{W}_k \hat{T}_k^{-2}\}\sigma_{min}^2[\mathcal{R}(0)]}{C_{\max}} \cdot \min\left\{\frac{\hat{u}_k}{M_k}\right\} P_{\min}. \tag{12}$$

Alternatively, the gains may satisfy a more stringent bound expressed solely in terms of network parameters:

$$\ell_{max} \leq \frac{4N_{min}^3 \sigma_{min}^2[\mathcal{R}(0)]}{mT_{max}^4 C_{max}^3}. \tag{13}$$

□

B. Design stabilizing PI

A PI AQM computes marking probability based on the difference between instantaneous queue length and a operator-defined set-point. Due to the integral element, the equilibrium queue length matches the set-point. The form of the PI AQM is

$$d_i(s) = \frac{l_i(s + z_i)}{s}. \tag{14}$$

Stabilizing parameters are described in the next result.

Proposition 2 (See Appendix E for proof): *Consider an* m*-link,* n*-source TCP/PI network described by* (3) − (5) *and* (14)*. Let the PI parameters be chosen as:* $l_i \in (\ell_{min},\ \ell_{max})$ *and* $z_i \in (Z_{\min},\ \omega_0)$ *where*

$$\omega_0 = 0.1 \min\left\{\frac{2}{\hat{W}_k \tau_k}, \frac{1}{\tau_k}, \lambda_{\min}\right\}.$$

Then, this network is locally stable if

$$1 - \frac{Z_{\min}\ell_{min}}{Z_{\max}\ell_{max}} \leq \frac{\sigma_{min}^2[\tilde{\mathcal{R}}(0)]}{2} \frac{\min\{\frac{M_k}{\hat{u}_k}\}}{\max\{\frac{M_k}{\hat{u}_k}\}} \tag{15}$$

and

$$\ell_{max} \leq \frac{\min\{N_k \hat{W}_k \hat{T}_k^{-2}\}\sigma_{min}^2[\mathcal{R}(0)]}{\sqrt{2}C_{\max}\left(\max\{\frac{M_k}{2\hat{u}_k}\} + \frac{\sigma_{min}^2[\tilde{\mathcal{R}}(0)]}{2}\min\{\frac{M_k}{2\hat{u}_k}\}\right)}. \tag{16}$$

Alternatively, the gains may satisfy a more stringent bound expressed in terms of network parameters:

$$\ell_{max} \leq \frac{4\sqrt{2}N_{min}^3 \sigma_{min}^2[\mathcal{R}(0)]}{3mT_{max}^4 C_{max}^3}. \tag{17}$$

□

Remarks: (i) In (13) and (17), we express AQM design rules as the function of network parameters, such as, link numbers and capacities, round-trip times, number of sessions and routing structure. This is appealing

because network administers do not need to estimate the equilibrium point to design stable AQMs.
(ii) It is seen in the above Propositions that the minimal singular value of routing matrix $\mathcal{R}$ affects network stability. This is an interesting observation because it implies that the routing structure itself actually affects network gain. We already recognize that small time delays and small link capacities increase the AQM gain margin. Now we show that larger routing gain increase these margins. To our knowledge, this is the first time connection has been made between routing and congestion control. We discuss this topic further in the next section. We now illustrate the PI AQM design rules for a three-source, two-link network.

Example (three-source, two-link network): Consider the network in Figure 2 where the routing matrix, link capacities, transmission delays, reference queue lengths and TCP loads are described by

$$\begin{aligned} \mathcal{R}(0) &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}; \; C = \begin{bmatrix} 3750 \\ 4250 \end{bmatrix} pkts/s; \\ T_p &= \begin{bmatrix} 0.05 \\ 0.06 \\ 0.07 \end{bmatrix} sec; \; q_{ref} = \begin{bmatrix} 150 \\ 180 \end{bmatrix} pkts/s; \\ N &= \begin{bmatrix} 30 \\ 40 \\ 50 \end{bmatrix} flows. \end{aligned}$$

Assuming a PI AQM, the equilibrium marking probabilities, window sizes, round-trip time and aggregate marking probabilities are computed to be

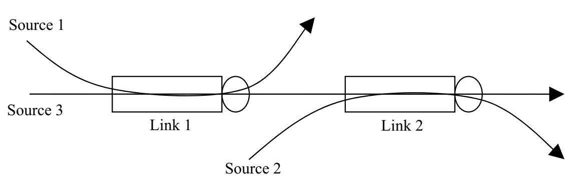

Fig. 2. The 3-source-2-link network in the example.

$$\begin{aligned} \hat{p} &= \begin{bmatrix} 0.0469 \\ 0.0427 \end{bmatrix}; \; \hat{W} = \begin{bmatrix} 6.5274 \\ 6.8470 \\ 4.7245 \end{bmatrix} pkts; \\ \hat{T} &= \begin{bmatrix} 0.09 \\ 0.1024 \\ 0.1524 \end{bmatrix} sec; \; \hat{u} = \begin{bmatrix} 0.0469 \\ 0.0427 \\ 0.0896 \end{bmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} F(s) &= diag\left\{ \frac{36.3e^{-0.09s}}{s+3.4}, \; \frac{33.4e^{-0.1s}}{s+2.85}, \; \frac{31e^{-0.15s}}{s+2.8} \right\}; \\ G(s) &= diag\left\{ s+6.28, \; s+11.4 \right\}. \end{aligned}$$

Using Proposition 2, we can compute $\omega_0 = 0.28$, $\ell_{max} \leq 0.11$ and $\frac{Z_{\min} Z_{\max}}{Z_{\max} \ell_{max}} \geq 0.72$. Selecting $Z_{\max} = 0.2$ and $\ell_{max} = 0.1$, the PI AQMs are chosen to be

$$D(s) = diag\left\{ \frac{0.1(s+0.2)}{s}, \; \frac{0.09(s+0.18)}{s} \right\}.$$

With $d(j\omega) = \frac{0.1(j\omega+0.2)}{j\omega}$, the set $S(\omega)$ is illustrated in Figure 3, showing that the network is locally stable. The two solid lines inside the boundary are the non-zero eigenvalues of $L(j\omega)$.

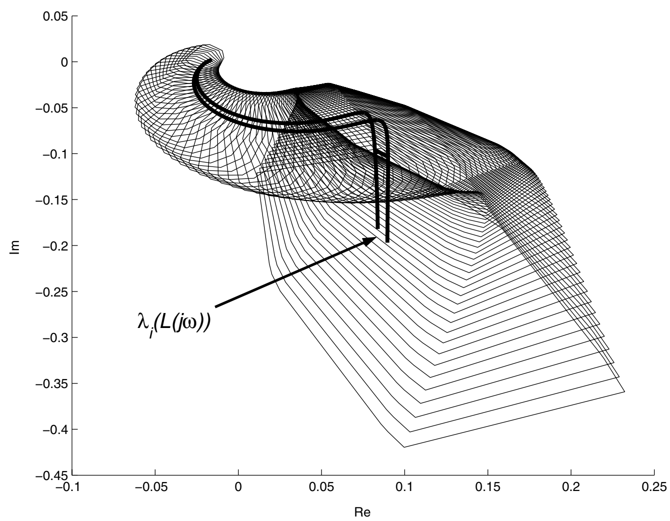

Fig. 3. $\lambda_i(L(j\omega))$ and their boundaries $S(\omega)$ for $\omega \in [0.1, 10]$.

IV. Impact of Routing on Congestion Control

In this section we explore the impact that routing has on congestion control through illustrative simulations, analysis of $\sigma_{min}[\mathcal{R}(0)]$'s range and discussion on the physical explanation of this phenomenon. For ease of exposition we will sometimes refer to $\sigma_{min}[\mathcal{R}(0)]$ as the "minimal gain" of the routing matrix (or route) $\mathcal{R}(0)$.

A. Illustrative `ns` simulations

To confirm and illustrate the impact that the minimal gain of $\mathcal{R}(0)$ has on congestion control, we consider a network with six sources, each comprised of an aggregate of $N = 100$ flows (ftp sessions), and 3 congested links. We consider two possible interconnections of these sources and links as shown in Figures 4 and 5.

The routing matrices describing these two topologies, $\mathcal{R}(0)_{big}$ and $\mathcal{R}(0)_{small}$ respectively, have quite different gains. Indeed, the topology in Figure 4 has routing matrix

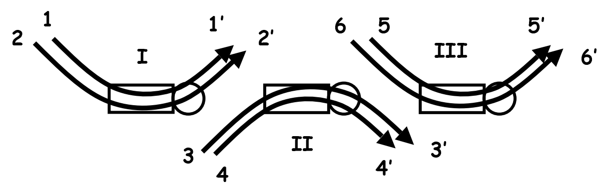

Fig. 4. Network topology associated with $\mathcal{R}(0)_{big}$ and $\sigma^2_{min}[\mathcal{R}(0)_{big}]$ = 2.

with minimal gain $\sigma^2_{min}[\mathcal{R}(0)_{big}] = 2$ where

$$\mathcal{R}(0)_{big} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix};$$

while the interconnection in Figure 5 is described by

$$\mathcal{R}(0)_{small} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

with gain $\sigma^2_{min}[\mathcal{R}(0)_{small}] = 0.37$. We now apply the

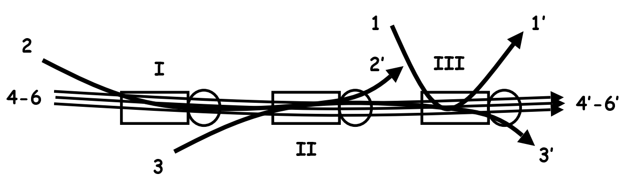

Fig. 5. Network topology associated with $\mathcal{R}(0)_{small}$ and $\sigma^2_{min}[\mathcal{R}(0)_{small}]$ = 0.37.

same AQM rule to each and observe its performance in regulating the three queue lengths. From (17), one would expect that the topology with the smaller $\mathcal{R}(0)$ to have smaller stability margins and display oscillatory behavior. To verify this, we conduct `ns` simulations with link capacities $c_1 = 3000$, $c_2 = 4000$ and $c_3 = 5000$ pkts/sec and propagation delays $\tau_{p1}, \tau_{p2}, \tau_{p3}$ taken uniformly in the range $(50, 200)$ms. The PI AQMs are described by

$$k_i(s) = \frac{1}{c_i}\left(2.3 \times 10^{-4} + \frac{3.1 \times 10^{-7}}{s}\right)$$

for $i = 1, 2, 3$. These were discretized at a sampling frequency of 160Hz for implementation in `ns`. The resulting responses for the three queue lengths I - III

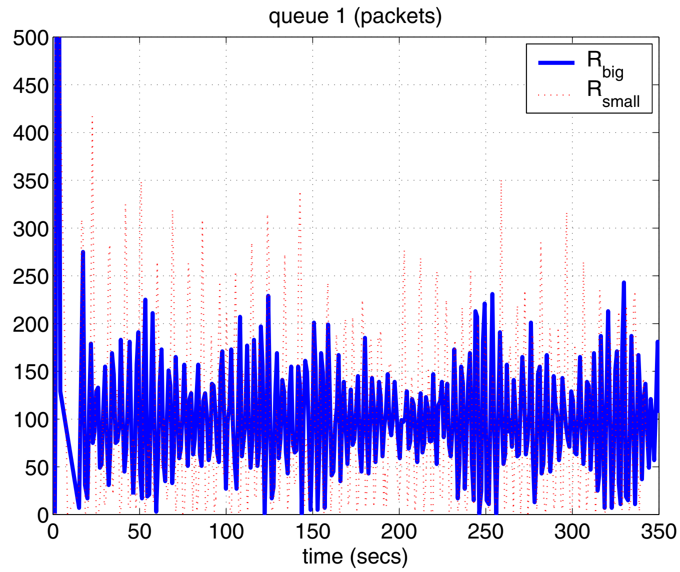

Fig. 6. Queue length response at buffer I.

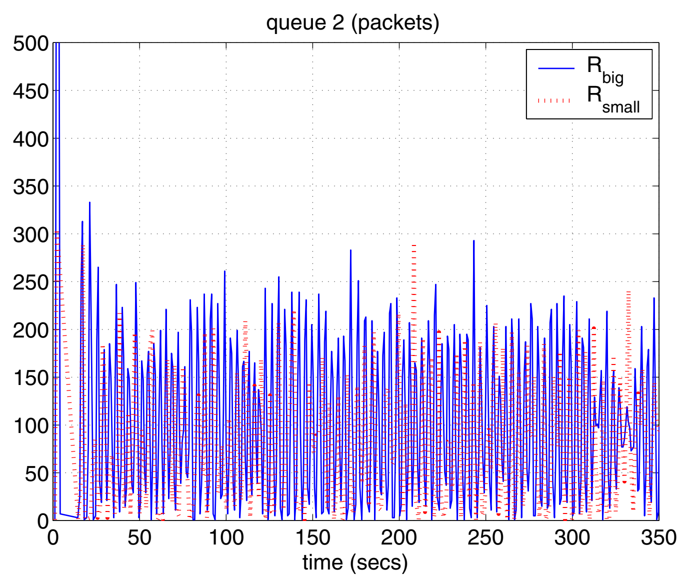

Fig. 7. Queue length response at buffer II.

are shown in Figures 6 – 8. The queue length responses for the case of $\mathcal{R}(0)_{small}$ exhibit larger amplitude oscillations. This confirms (17) which predicts that gain margins are proportional to the minimal gain of $\mathcal{R}(0)$, and, that the network is "closer" to instability as expressed by the oscillations. Clearly, if we can anticipate that a topology would be described by a routing matrix with small minimal gain, as in $\sigma_{min}[\mathcal{R}(0)_{small}]$, then one can compensate with appropriate AQM tuning. However, in the absence of such knowledge, we face the task of tuning AQM's to deal with a host of topologies described

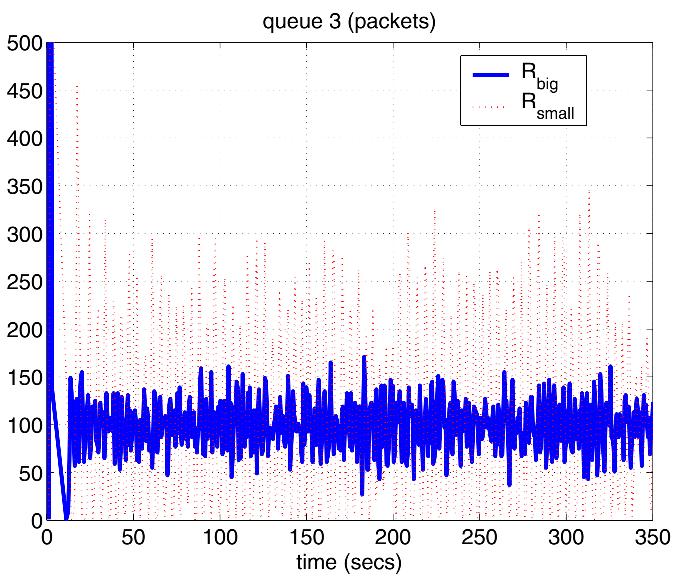

Fig. 8. Queue length response at buffer III.

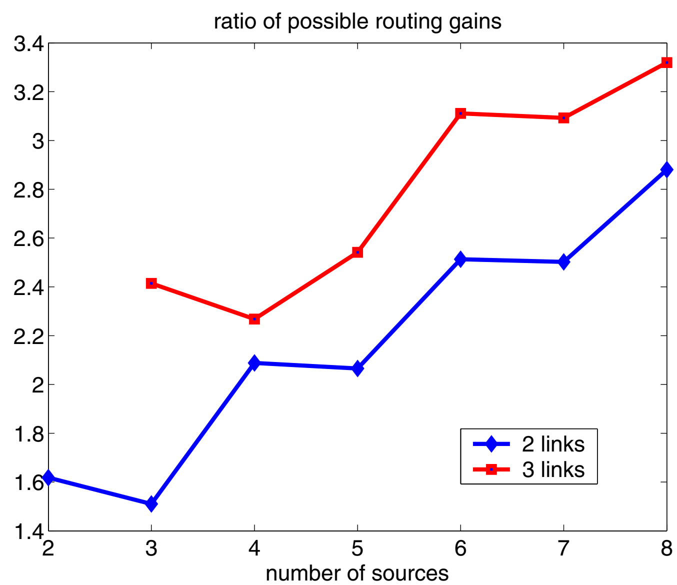

Fig. 9. Ratio of possible routing gains vs. # of sources and # of congested links.

by routing matrices with a (possibly) large range of gains. This possibility begs an obvious question related to the range of gains of routes.

B. What is the range of routing gains?

The range of routing gains may be significant as illustrated in Figure 9 where we plot an estimate of the ratio $\max \sigma_{min}[\mathcal{R}(0)]/\min \sigma_{min}[\mathcal{R}(0)]$, where the min and max are taken over the binary matrices. This plot was generated experimentally by enumerating all possible full-rank binary matrices (of dimension # links $\times$ # sources). This plot says that for 3 congested links traversed by 5 sources, the range of minimal route gains is approximately 2.6. It appears from this experiment that range of routing gains is proportional to the number of sources and links. To examine this more closely consider the set of *routing matrices*

$$\mathcal{A} = \{\mathcal{R}(0) \in \mathbf{R^{m \times n}} : \mathcal{R}(0) \text{ binary and full rank}\}.$$

With $int[n/m]$ denoting the integer part of n/m, the following result provides a lower bound to the range of gains.

Proposition 3: *Given $m \leq n$,*

$$\frac{\max_{\mathcal{A}} \sigma_{min}[\mathcal{R}(0)]}{\min_{\mathcal{A}} \sigma_{min}[\mathcal{R}(0)]} \geq \sqrt{m}\,\sqrt{int[n/m]}. \qquad (18)$$

Proof: First, we will prove that $\min_{\mathcal{A}} \sigma_{min}[\mathcal{R}(0)] \leq 1/\sqrt{m}$. To this end, consider the specific routing matrix

$$\tilde{\mathcal{R}}(0) = [X \;\; 0_{m \times n-m}]$$

where $X \in \mathbf{R}^{m \times m}$ is an identity matrix with $1's$ on its superdiagonal; e.g., for $m = 3$,

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Given $x \in \mathbf{R}^n$, partition $x = [y^T \;\; z^T]^T$ with $y \in \mathbf{R}^m$. Then,

$$\begin{aligned} \min_{\mathcal{A}} \sigma_{min}[\mathcal{R}(0)] &\leq \sigma_{min}[\tilde{\mathcal{R}}(0)] \\ &\triangleq \max_{\substack{S \subset \mathbf{R}^n \\ dim S = \mathbf{R}^m}} \min_{\substack{x \in S \\ \|x\|_2 = 1}} \|\tilde{A}x\|_2 \\ &= \min_{\substack{y \in \mathbf{R}^m \\ \|y\|_2 = 1}} \|Xy\|_2. \end{aligned}$$

Now, with

$$\tilde{y} = \frac{1}{\sqrt{m}}[\,1 \;\; -1 \;\; 1 \;\; -1 \;\; \ldots \;\; \pm 1]^T \in \mathbf{R}^m$$

we have $\|\tilde{y}\|_2 = 1$ and

$$X\tilde{y} = \frac{1}{\sqrt{m}}[\;0 \;\; 0 \;\; \ldots \;\; 0 \;\; \pm 1\;]^T.$$

It is easy to see that $\|Xy\|_2 \geq \|X\tilde{y}\|_2$ for all $\|y\|_2 = 1$. Thus,

$$\min_{\mathcal{A}} \sigma_{min}[\mathcal{R}(0)] \leq \min_{\substack{y \in \mathbf{R}^m \\ \|y\|_2 = 1}} \|Xy\|_2 = \|X\tilde{y}\|_2 = 1/\sqrt{m}.$$

To finish the proof we will now show that $\max_{\mathcal{A}} \sigma_{min}[\mathcal{R}(0)] \geq \sqrt{int[n/m]}$. We again consider a specific form of routing matrix, this time given by

$$\hat{\mathcal{R}}(0) = [\ I_m \quad I_m \quad \ldots \quad I_m \quad B\ ]$$

where B is the any $m \times k$ matrix with $k < m$. For example, for $m = 2, n = 5$

$$\hat{\mathcal{R}}(0) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

where we have taken $B = [0 \ \ 1]^T$. Since

$$\hat{\mathcal{R}}(0)\hat{\mathcal{R}}(0)^T = I_m + I_m + \ldots + I_m + BB^T,$$

then

$$\begin{aligned} \lambda_{min}[\hat{\mathcal{R}}(0)\hat{\mathcal{R}}(0)^T] &= 1 + 1 + \ldots + 1 + \lambda_{min}[BB^T] \\ &= 1 + 1 + \ldots + 1 \\ &= int\,[n/m]. \end{aligned}$$

Thus,

$$\max_{\mathcal{A}} \sigma_{min}[\mathcal{R}(0)] \geq \sigma_{min}[\hat{\mathcal{R}}(0)] = \sqrt{int\,[n/m]}.$$

□

From the gain margin bound (17), we see that network robustness scales inversely with the routing matrix gain range (18). The proposition's results would then indicate that robustness is decreased with increasing number of congested routers m and sources n. The example in the previous subsection dealt with a six-source, three-link network having gain range of $\sqrt{2/0.37} \approx 2.3$ and showed how the quality of buffer regulation was affected by routes of differing gains.[3] Such effects may be even more pronounced with increased complexity of network topologies.

C. Physical Explanation of the gain of routing

The reason why $\sigma_{min}[\mathcal{R}(0)]$ directly affects congestion control comes from the fact that TCP source rates are influenced by round-trip times, and, these round-trip times are affected by queueing delays. Hence, variations in (non-zero) buffer lengths affect the flows that fill these buffers. So, there is feedback path from buffer lengths to buffer inputs. Now, the queueing delays are communicated to the TCP-sources through the Internet, and through the routing matrix. Thus, a model of this interaction is a feedback loop with buffers in the forward path, and routing matrix in the feedback path. As the gain of the routing matrix diminishes, this feedback loop behaves more closely to an open loop; i.e., simply open-loop buffers. An integrator is a linear approximation to an open-loop buffer. In essence, when a real pole in a feedback system is being shifted toward the origin (i.e., toward becoming an integrator), the stability margin is being reduced. It is the presence of this diminished routing matrix gain that tends to destabilize the congestion control feedback system.

[3] Compare this gain range of 2.3 against the theoretical lower bound on gain range: $\sqrt{m\,\mathrm{int}[n/m]} = \sqrt{3 \cdot 2} = 2.45$, and the experimentally-derived range value of ≈ 3.1 shown in Figure 9 corresponding to the six-source, three-link point on the graph.

V. Conclusions

In this paper we have derived new stability conditions that are applicable to today's Internet described by TCP-controlled sources and buffer-based AQMs. The main result, found in Theorem 1, gives a stability condition that describes the interplay between the gains of the TCP sources and queuing delays with the AQM gains. This condition is directly applicable to networks with rate-based AQMs and reduces to typical stability results for network with a single congested link. Our stability condition can be used to specify parameters for stabilizing RED and PI AQMs as explicitly stated in Propositions 1 and 2. Gain inequalities (13) and (17) are particularly appealing as they state design rules *explicitly* in terms of network parameters rather than equilibrium values. These inequalities serve another purpose by providing *gain margin* estimates for buffer-based AQM networks. In this context, (13) and (17) reveal that a so-called *gain of routing* can play a role in stability robustness - a phenomenon absent in rate-based AQMs. In Section 4 we examined this impact and showed that the *range* of routing gains is proportional to the number of sources and links. This raises a number of interesting questions:

- In practice how many congested links do sources encounter?
- The lower bound on gain range considered all possible routing matrices. Is this a reasonable assumption, or do present routing algorithms result in specific pattern of routing matrices $\mathcal{R}(0)$? If so, what is the gain range for this class.
- How does shortest-path routing impact congestion control?
- How do decentralized routing decisions impact global congestion control concerns?
- How does one analyze stability of networks with mixed RED and PI AQMs?

Future research is aimed at addressing some of these questions.

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Appendix A
Loop Transfer Function $L(s)$

Combining (3) - (5) gives

$$L(s) = \tilde{F}\mathcal{R}(-s)^T DC^{-1}(sI+\Omega)^{-1}\mathcal{R}(s)N\hat{T}^{-1}.$$

Since $C^{-\frac{1}{2}}\Omega C^{\frac{1}{2}}$ is positive definite, then $C^{-\frac{1}{2}}\Omega C^{\frac{1}{2}} = V diag\{\lambda_i\}V^H$ where λ_i are the eigenvalues of $C^{-\frac{1}{2}}\Omega C^{\frac{1}{2}}$ and V is unitary. Consequently,

$$\begin{aligned} C^{-1}(sI+\Omega)^{-1} &= C^{-\frac{1}{2}}(sI + C^{-\frac{1}{2}}\Omega C^{\frac{1}{2}})^{-1}C^{-\frac{1}{2}} \\ &= C^{-\frac{1}{2}}(VGV^H)^{-1}C^{-\frac{1}{2}} \end{aligned}$$

where $G(s) = sI + diag\{\lambda_i\}$.

Appendix B
Details for proof of Theorem 1

Claim 1:

$$\lambda_{\min} \geq \frac{\min\{N_k\hat{W}_k\hat{T}_k^{-2}\}\sigma^2_{min}[\mathcal{R}(0)]}{C_{\max}}.$$

Proof of Claim 1: We have

$$\begin{aligned} \lambda_{\min} &= \min\{\lambda(C^{-\frac{1}{2}}\Omega C^{\frac{1}{2}})\} \\ &= \frac{1}{\rho(C(\mathcal{R}(0)N\hat{W}\hat{T}^{-2}\mathcal{R}(0)^T)^{-1})} \\ &\geq \frac{1}{C_{max}\min\{\lambda(\mathcal{R}(0)N\hat{W}\hat{T}^{-2}\mathcal{R}(0)^T)\}}. \end{aligned}$$

Since,

$$\begin{aligned} \min\{\lambda(\mathcal{R}(0)N\hat{W}\hat{T}^{-2}\mathcal{R}(0)^T)\} &\geq \\ &\min\{N_k\hat{W}_k\hat{T}_k^{-2}\}\sigma^2_{min}[\mathcal{R}(0)] \end{aligned}$$

we get

$$\lambda_{\min} \geq \frac{\min\{N_k\hat{W}_k\hat{T}_k^{-2}\}\min\{\lambda(\mathcal{R}(0)\mathcal{R}(0)^T)\}}{C_{\max}}.$$

Claim 2: $\|\tilde{\mathcal{R}}\tilde{\mathcal{R}}^H\|_2 \leq 1$

Proof of Claim 2:

$$\begin{aligned} \sigma^2_{max}[\mathcal{R}] &= \rho(\tilde{\mathcal{R}}\tilde{\mathcal{R}}^H) \\ &= \rho(C^{-1}\mathcal{R}N\hat{W}\hat{T}^{-1}M^{-1}\mathcal{R}^H) \\ &\leq \|C^{-1}\mathcal{R}N\hat{W}\hat{T}^{-1}\|_\infty \|M^{-1}\mathcal{R}^H\|_\infty \leq 1. \end{aligned}$$

Claim 3: For $d \in \mathbf{C}$,

$$\begin{aligned} \mathcal{F}\left(\tilde{\mathcal{R}}F\tilde{\mathcal{R}}^H D\right) &\subseteq \\ & Co\{\sigma^2_{min}[\tilde{\mathcal{R}}], \sigma^2_{max}[\mathcal{R}]\}Co\{df_i\} \\ &\quad + disc(\sigma^2_{max}[\mathcal{R}]\,\|F\|_2\,\|D-dI\|_2). \end{aligned}$$

Proof:

$$\begin{aligned} \mathcal{F}\left(\tilde{\mathcal{R}}F\tilde{\mathcal{R}}^H D\right) &= \mathcal{F}\left(\tilde{\mathcal{R}}F\tilde{\mathcal{R}}^H(dI + D - dI)\right) \\ &\subseteq \mathcal{F}\left(\tilde{\mathcal{R}}Fd\tilde{\mathcal{R}}^H\right) \\ &\quad + \mathcal{F}\left(\tilde{\mathcal{R}}F\tilde{\mathcal{R}}^H(D-dI)\right) \end{aligned}$$

Also,

$$\begin{aligned} \mathcal{F}\left(\tilde{\mathcal{R}}Fd\tilde{\mathcal{R}}^H\right) &= \{x^H\tilde{\mathcal{R}}Fd\tilde{\mathcal{R}}^H x : x^H x = 1\} \\ &\subset \{\sum |y_i|^2 : y = \tilde{\mathcal{R}}^H x, x^H x = 1\} \\ &\quad \cdot Co\{df_i\} \\ &\subset Co\{\sigma^2_{min}[\tilde{\mathcal{R}}], \sigma^2_{max}[\tilde{\mathcal{R}}]\}Co\{df_i\}. \end{aligned}$$

Since

$$\begin{aligned} |\mathcal{F}\left(\tilde{\mathcal{R}}F\tilde{\mathcal{R}}^H(D-dI)\right)| &\leq \|\tilde{\mathcal{R}}F\tilde{\mathcal{R}}^H(D-dI)\|_2 \\ &\leq \sigma^2_{max}[\tilde{\mathcal{R}}]\|F\|_2\|D-dI\|_2, \end{aligned}$$

then

$$\mathcal{F}\left(\tilde{\mathcal{R}}F\tilde{\mathcal{R}}^H(D-dI)\right) \subseteq disc(\sigma^2_{max}[\tilde{\mathcal{R}}]\,\|F\|_2\,\|D-dI\|_2).$$

Appendix C
Proof of Corollary 1

The eigenvalues of $L(s)$ satisfy:

$$\begin{aligned} \lambda(L(s)) &\subseteq \mathcal{F}\left(\tilde{\mathcal{R}}F(j\omega)\tilde{\mathcal{R}}^H(D(j\omega)^{-1})^{-1}\right) \\ &\subseteq \mathcal{F}\left(\tilde{\mathcal{R}}F(j\omega)\tilde{\mathcal{R}}^H I\right)/\mathcal{F}(D(j\omega)^{-1}). \end{aligned}$$

Since $0 \notin Co\,(1/d_k(j\omega))$, then $0 \notin \mathcal{F}(D(j\omega)^{-1})$. For $b \in \mathbf{C}$

$$\begin{aligned} \mathcal{F}(\tilde{\mathcal{R}}F\tilde{\mathcal{R}}^H I) &\subseteq \\ & Co\{\sigma^2_{min}[\tilde{\mathcal{R}}], 1\}Co\{bf_j\} + disc(\|F\|_2\,\|I-bI\|_2) \end{aligned}$$

so that

$$\mathcal{F}(\tilde{\mathcal{R}}F\tilde{\mathcal{R}}^H I) \subseteq Co\{\sigma^2_{min}[\tilde{\mathcal{R}}], 1\}Co\{f_j\}$$

and

$$\lambda(L(s)) \subseteq Co\{\sigma^2_{min}[\tilde{\mathcal{R}}], 1\}Co\{f_j\}/Co\,\{1/d_k(j\omega)\}\,.$$

Appendix D
Proof of Proposition 1

To apply Theorem 1, we select $d_\omega \equiv 0$. We will show

$$\min\{|g_i(j\omega)|\} > \max\{|f_k(j\omega)|\}\max\{|d_i(j\omega)|\} \quad (19)$$

which in turn will imply that $\frac{disc(\|F\|_2\|D\|_2)}{Co\{g_i\}} \subset Re(s) > -1$. With the lower bound given on $\lambda_{\min}$ in Appendix B, (12) implies that

$$\ell_{max} \le 2\lambda_{\min}\min\{\frac{\hat{u}_k}{M_k}\}P_{\min}. \quad (20)$$

Because $l_i < \ell_{max}$ and $p_i \ge P_{\min}$, then,

$$\max\{|d_i(0)|\} < \frac{\ell_{max}}{P_{\min}}.$$

Therefore, applying (20), we get

$$\max\{|d_i(0)|\} < 2\lambda_{\min}\min\{\frac{\hat{u}_k}{M_k}\} = \frac{\lambda_{\min}}{\max\{\frac{M_k}{2\hat{u}_k}\}},$$

which means

$$\max\{|f_k(0)|\}\max\{|d_i(0)|\} < \min\{|g_i(0)|\}. \quad (21)$$

Because $\min\{|g_i(j\omega)|\}$ is an increasing function, $\max\{|f_k(j\omega)|\}$ and $\max\{|d_i(j\omega)|\}$ are decreasing. Thus, (21) implies (19). Next, we prove (13). Because

$$\min\{\frac{N_k\hat{W}_k}{\hat{T}_k^2}\} \ge \frac{N_{\min}\hat{W}_{\min}}{\hat{T}_{\max}^2} > \frac{N_{\min}}{\hat{T}_{\max}^2} \quad (22)$$

and

$$\min\{\frac{\hat{u}_k}{M_k}\} \ge \frac{\hat{u}_{\min}}{M_k} \ge \frac{\hat{u}_{\min}}{m}.$$

and Appendix F, the RHS of (12) is greater than

$$\frac{4N_{\min}^3\min\{\lambda(\mathcal{R}(0)\mathcal{R}(0)^T)\}}{mT_{\max}^4C_{\max}^3}P_{\min}.$$

Thus if (13) holds, (12) holds.

Appendix E
Proof of Proposition 2

First we select

$$d(j\omega) = \frac{\ell_{max}(j\omega + Z_{\max})}{j\omega}.$$

Thus, for each $l_i < \ell_{max}$,

$$\begin{aligned}|\frac{d_i(j\omega)}{d(j\omega)} - 1| &= \\ \left(\frac{\ell_{max} - l_i}{\ell_{max}}\right)&\left|\frac{j\omega + Z_{\max}\left(\frac{l_i\frac{z_i}{z_{\max}} - \ell_{max}}{l_i - \ell_{max}}\right)}{j\omega + Z_{\max}}\right|. \end{aligned} \quad (23)$$

Because $l_i < \ell_{max}$ and $z_i < Z_{\max}$,

$$\left(\frac{l_i\frac{z_i}{z_{\max}} - \ell_{max}}{l_i - \ell_{max}}\right) > 1.$$

Then, for all $\omega \ge 0$,

$$\left|\frac{j\omega + Z_{\max}\left(\frac{l_i\frac{z_i}{z_{\max}} - \ell_{max}}{l_i - \ell_{max}}\right)}{j\omega + Z_{\max}}\right| < \left(\frac{l_i\frac{z_i}{z_{\max}} - \ell_{max}}{l_i - \ell_{max}}\right). \quad (24)$$

Substituting (24) into (23) yields

$$\left|\frac{d_i(j\omega)}{d(j\omega)} - 1\right| < \frac{\ell_{max} - l_i\frac{z_i}{Z_{\max}}}{\ell_{max}}.$$

Therefore, for all $l_i \in (\ell_{min} \;\; \ell_{max})$ and $\forall z_i \in (Z_{\min}\; Z_{\max})$,

$$\left|\frac{d_i(j\omega)}{d(j\omega)} - 1\right| < 1 - \frac{Z_{\min}\ell_{min}}{Z_{\max}\ell_{max}}.$$

Case 1 ($\omega < \omega_0$): If $\omega < \omega_0$, the set $Co\{f_k\}$ is around the positive real axis. So, the smallest angle of elements in set

$$Co\{\sigma_{min}^2[\tilde{\mathcal{R}}], 1\}Co\{f_k\} + disc(\|F\|_2\|\frac{1}{d}D - I\|)$$

is approximately larger than $-\pi/6$ by (15). Also, the angle of $d(j\omega)$ is always larger than $-\pi/2$. So, $\omega < \omega_0$, the smallest angle of elements in set

$$\begin{aligned}&S(f,d) \equiv \\ &\quad Co\{\sigma_{min}^2[\tilde{\mathcal{R}}], 1\}Co\{f_k\} + disc(\|F\|_2\|\frac{1}{d}D - I\|)\end{aligned}$$

is larger than $-2\pi/3$. However, the largest angle of elements in set

$$(-\infty \;\; -1]Co\{g_i\} \equiv S(g)$$

will be less than $-\pi + 0.1$ when $\omega < \omega_0$. So, (7) in Theorem 1 is satisfied for $\omega \in (0 \;\; \omega_0)$.

Case 2 ($\omega \ge \omega_0$): If $\omega \ge \omega_0$, then,

$$\begin{aligned}|d(j\omega)| \le |d(j\omega_0)| &= \ell_{max}\sqrt{1 + \left(\frac{Z_{\max}}{\omega_0}\right)^2} \\ &\le \sqrt{2}\ell_{max},\end{aligned}$$

$$\begin{aligned}|f_k(j\omega)| \le |f_k(j\omega_0)| &= \frac{1}{\sqrt{\omega_0^2 + \left(\frac{2}{\hat{W}_k\tau_k}\right)^2}}\frac{M_k}{\hat{W}_k\tau_k\hat{u}_k} \\ &< \frac{M_k}{2\hat{u}_k}\end{aligned}$$

and

$$|g_i(j\omega)| > |g_i(j\omega_0)| = \sqrt{\omega_0^2 + \lambda_i^2} > \lambda_i.$$

So, the largest amplitude of elements in set $S(f,d)$ is no more than

$$\begin{aligned} |S(f,d)| &\leq \sqrt{2}\ell_{max} \max\{\frac{M_k}{2\hat{u}_k}\} \left(2 - \frac{Z_{\min}\ell_{min}}{Z_{\max}\ell_{max}}\right) \\ &\leq \sqrt{2}\ell_{max}(\max\{\frac{M_k}{2\hat{u}_k}\} + \frac{\sigma_{min}^2[\tilde{\mathcal{R}}]\min\{\frac{M_k}{2\hat{u}_k}\}}{2}). \end{aligned}$$

Thus, by Appendix B, (16) implies that

$$\ell_{max} \leq \frac{\lambda_{\min}}{\sqrt{2}\left(\max\{\frac{M_k}{2\hat{u}_k}\} + \frac{\sigma_{min}^2[\tilde{\mathcal{R}}]}{2}\min\{\frac{M_k}{2\hat{u}_k}\}\right)},$$

which means that the maximal amplitude of elements in set $S(f,d)$ is less than the smallest amplitude of elements in set $S(g)$ when $\omega \geq \omega_0$. Thus, set $S(f,d)$ does not intersect set $S(g)$, so (7) in Theorem 1 is satisfied for $\omega \geq \omega_0$. Next, we prove (17). Because

$$\max\{\frac{M_k}{2\hat{u}_k}\} \leq \frac{m}{2\hat{u}_{\min}}$$

and

$$\frac{\sigma_{min}^2[\tilde{\mathcal{R}}]}{2}\min\{\frac{M_k}{2\hat{u}_k}\} \leq \frac{1}{2}\max\{\frac{M_k}{2\hat{u}_k}\}.$$

Thus

$$\max\{\frac{M_k}{2\hat{u}_k}\} + \frac{\sigma_{min}^2[\tilde{\mathcal{R}}]}{2}\min\{\frac{M_k}{2\hat{u}_k}\} \leq \frac{3m}{4\hat{u}_{\min}}.$$

So, the RHS of (16) is greater than

$$\frac{4\hat{u}_{\min}}{3m}\frac{\min\{N_k\hat{W}_k\hat{T}_k^{-2}\}\min\{\lambda(\mathcal{R}(0)\mathcal{R}(0)^T)\}}{\sqrt{2}C_{\max}}$$

From (22) and Appendix F, it is easy to show that above function is greater than

$$\frac{4\sqrt{2}N_{\min}^3\min\{\lambda(\mathcal{R}(0)\mathcal{R}(0)^T)\}}{3mT_{\max}^4C_{\max}^3}.$$

Now, (17) implies that (16) holds.

Appendix F
Lower bound on $\hat{u}_{\min}$

Since the rate for each source is bounded by link capacities, then,

$$\frac{N_i\hat{W}_i}{\hat{T}_i} \leq C_{\max}$$

which implies that

$$\hat{W}_{\max} \leq \frac{C_{\max}\hat{T}_{\max}}{N_{\min}}.$$

Because

$$\hat{W}_{\max} = \sqrt{\frac{2}{\hat{u}_{\min}}},$$

yields

$$\hat{u}_{\min} \geq \frac{2N_{\min}^2}{\hat{T}_{\max}^2C_{\max}^2}$$

References

[1] L. Le, J. Aikat, K. Jeffay and F. D. Smith, "The Effects of Active Queue Management on Web Performance," *ACM SIGCOMM* 2003.

[2] S. Floyd and V. Jacobson, "Random Early Detect Gateways for Congestion Avoidance," *IEEE/ACM Transactions on Networking*, Vol.1, August 1997.

[3] S. Kunniyur and R. Srikant, "Analysis and Design of an Adaptive Virtual Queue Algorithm for Active Queue Management," *Proceedings of ACM SIGCOMM*, 2001.

[4] C. V. Hollot, V. Misra, D. Towsley and W. B. Gong, "Analysis and Design of Controllers for AQM Routers Supporting TCP Flows," *IEEE Transactions on Automatic Control,* June 2002.

[5] S. Athuraliya, D. E. Lapsley and S. H. Low, "Random Early Marking for Internet Congestion Control," *Proceedings of IEEE GLOBECOM*, 1999.

[6] F. P. Kelly, A. K. Maulloo and D. Tan, "Rate Control for Communication Networks: Shadow Prices, Proportional Fairness and Stability," *Journal of the Operational Research Society,* 49, 237-252, 1998.

[7] R. Johari and D. Tan, "End-to-End Congestion Control for the Internet: Delays and Stability," *IEEE/ACM Transactions on Networking,* 9, 818-832, 2001.

[8] V. Misra, W. B. Gong, and D. Towsley, "Fluid-based Analysis of a Network of AQM Routers Supporting TCP Flows with an Application to RED," *Proceedings of ACM/SIGCOMM*, 2000.

[9] L. Massoulie, "Stability of Distributed Dongestion Dontrol with Heterogeneous Feedback Delays," *IEEE Transactions on Automatic Control,* 47, 895-902, 2002.

[10] G. Vinnicombe, "On the Stability of Networks Operating TCP-Like Congestion Control," Proceedings of the *IFAC World Congress*, Barcelona, 2002.

[11] G. Vinnicombe, "Robust Congestion Control for Internet," available at http://www-control.eng.cam.ac.uk/gv, 2002.

[12] S. Kunniyur and R. Srikant, "Stable, Scalable, Fair Congestion Control and AQM Schemes that Achieve High Utilization in the Internet," *CISS*, 2002.

[13] F. Paganini, J. Doyle and S. Low, "Scalable Laws for Stable Network Congestion Control," Proceedings of the *Conference on Decision and Control*, 2001.

[14] C. A. Deseor and Y. T. Yang, "On the Generalized Nyquist Stability Criterion," *IEEE Transactions on Automatic Control*, 25:187-196, 1980.

[15] R. A. Horn and C. R. Johnson, "Topics in Matrix Analysis," *Cambridge University Press*, 1991.