Limit Behavior of ECN/RED Gateways Under a Large Number of TCP Flows

Peerapol Tinnakornsrisuphap Armand M. Makowski
Department of Electrical and Computer Engineering and the Institute for Systems Research
University of Maryland College Park
College Park, Maryland 20742
Email: peerapol@eng.umd.edu, armand@isr.umd.edu

Abstract— **We consider a stochastic model of an ECN/RED gateway with competing TCP sources sharing the capacity. As the number of competing flows becomes large, the queue behavior at the gateway can be described by a two-dimensional recursion and the throughput behavior of individual TCP flows becomes asymptotically independent. The steady-state regime of the limiting behavior can be calculated from a well-known TCP throughput model with fixed loss probability. In addition, a Central Limit Theorem is presented, yielding insight into the relationship between the queue fluctuation and the marking probability function. We confirm the results by simulations and discuss their implications for network dimensioning.**

I. Introduction

One of the key mechanisms for operating the best-effort service Internet is the congestion control mechanism in TCP [1]. While there exist several variations of the basic TCP congestion control mechanism, they all have in common the *additive increase/multiplicative decrease* (AIMD) algorithm. This AIMD algorithm is a self-clocking feedback mechanism that enables TCP congestion-control to be robust under diverse conditions, but at the expense of introducing additional complexity into the behavior of network traffic. There has been a number of efforts aimed at gaining insights into this complex behavior, and by now the relationship between the throughput of a *single* TCP, its round-trip time and loss probability is fairly well understood [2] [3] [4] [5].

There remain, however, certain aspects of TCP that are not well understood and which cannot be analyzed readily with the models in these references. Among these outstanding issues we include the buffer behavior at a bottleneck router and the aggregate throughput that results from *many* TCP flows competing for the bandwidth of a link. While the earlier models could in principle be extended to answer some of these questions, the resulting models would not be scalable. Typically, with each TCP flow modeled in great details, the size of the state space for the model explodes when the number of flows becomes large, and the analysis then becomes intractable. Even numerical calculations or simulations of such models are very complicated and become computationally prohibitive, therefore providing no additional advantages over full-scale (system) simulation with existing simulation packages (e.g., NS [6]). To be sure, certain simplifying assumptions could be made, but it is not clear from the onset which irrelevant details can be omitted without reducing the predictive power of the model.

To make matters worse, recent developments in Active Queue Management (AQM) techniques have introduced additional complexity in transport protocols. The development of AQM was prompted by the observation that with simple Tail-Drop gateways, TCP congestion-control leads to undesirable behavior, i.e., global synchronization [7]. When several TCP flows compete for bandwidth in a Tail-Drop gateway, it has been observed experimentally that packets from many flows are usually discarded simultaneously, resulting in a poor utilization of the network. AQM algorithms such as Random Early Detection (RED) [8] and Explicit Congestion Notification (ECN) [9] have been proposed to help alleviate this problem by randomly dropping/marking packets with probability depending on queue size. This allows each TCP flow to react early to the growing congestion, thereby avoiding heavy congestion and preventing global synchronization. As can easily be imagined, the introduction of AQM further exacerbates the difficulty of understanding issues associated with buffer behavior and aggregate TCP traffic. Attempts have been made to model the interactions between TCP and AQM, but so far the analytically tractable models are either too crude or too simplistic as we now discuss:

Analytical efforts to model the interactions between TCP congestion-control and the bottleneck router/buffer usually involve various ad-hoc assumptions in order to render the analysis feasible. For example, Hollot et al. linearized the TCP mechanism and studied the system using a control-theoretic approach [10]. TCP being highly non-linear, the regime where such a linearization provides an accurate approximation is usually small. In [11], an analytical framework for multiple TCP flows sharing a RED gateway was developed under several potentially unrealistic assumptions. In [12], a simple analysis was carried out with TCP connections operating as Poisson processes under "slow" and "fast" rates.

Recently, there has been a growing interest in *macroscale* modeling of TCP flows, as opposed to microscale models where each TCP flow is modeled in detail. Macroscale TCP

This material is based upon work supported by the Space and Naval Warfare Systems Center – San Diego under Contract No. N66001-00-C-8063. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the Space and Naval Warfare Systems Center – San Diego.

models can be developed by systematically applying limit theorems to derive a limiting traffic model when the number of TCP flows is large. The potential benefits of doing so are three-fold. First, model simplification (with the promise of scalability) typically occurs when applying limit theorems, with irrelevant details filtered out without relying on ad-hoc assumptions. Second, limit theorems are central to the modern Theory of Probability, and as such have been the focus of a huge literature that contains a large number of results and techniques. Given this large body of knowledge, it is reasonable to expect the existence of suitable limit theorems (under very weak assumptions) which can be applied to the situation of interest. Finally, in the networking context, resource allocation problems are interesting only in networks operating at high utilization, e.g., when the number of users is large in relation to available resources. In such a scenario, the limit behavior will become increasingly more accurate as the number of users increases.

Limit theorems for a bottleneck queue under a large number of rate-controlled TCP-like flows have been recently considered by a number of authors [13], [14], [15]; a survey of the relevant literature can be found in [16]. While these models already suggest some of the preliminary results to be expected from aggregating a large number of TCP flows, they all lack the explicit window mechanism of TCP congestion control. In this paper, we incorporate this feature explicitly and follow the approach of [14] to model the ECN-capable TCP Reno congestion-control mechanism. We establish several asymptotics when the number of flows is large, namely a Law of Large Numbers for the aggregate traffic into the RED buffer and a basic limit theorem for the normalized queue size [Section IV]. We sharpen these results with a Central Limit Theorem (CLT) complement [Section VI]. These results help clarify the relationship between RED buffers and the marking-probability function. The CLT result can help in the network dimensioning problem by establishing a probability distribution on the buffer utilization in RED gateways. The concept of many source asymptotics used here has been well studied in ATM networks but is more difficult to obtain for IP networks because TCP traffic is feedback-based while ATM traffic is open-loop.

The paper is organized as follows. We first outline its main contributions in Section II. Then, in Section III the model is described in detail, and a first set of asymptotic results are presented in Section IV. Section V focuses on the calculations of the limiting normalized queue size and the average window size in steady state. Section VI contains a Central Limit Theorem complement to the asymptotic results of Section IV. Simulation results confirming the theoretical results are shown in Section VII. Applications to network dimensioning and to the design of marking probability function are discussed in Section VIII. Conclusions and future work are given in Section IX.

A word on the notation in use: Equivalence in law or in distribution between random variables (rvs) is denoted by $=_{st}$. The indicator function of an event A is simply $\mathbf{1}[A]$, and we use $\stackrel{P}{\to}_n$ (resp. $\Longrightarrow_n$) to denote convergence in probability (resp. weak convergence or convergence in distribution) with n going to infinity.

II. Contributions

This paper considers the scenario where N identical TCP traffic sources compete for bandwidth in the bottleneck router. The bottleneck capacity is NC and the router utilizes an ECN/RED marking scheme with the dropping/marking probability function scaling with N (in the sense of Assumption (A1) in Section IV). The main theoretical contributions of this paper can be summarized as follows:

(i) Theorem 1 shows that the dynamics of the queue at time t, denoted $Q^{(N)}(t)$, can be approximated by $Nq(t)$ with $q(t)$ determined via a simple deterministic recursion, which is independent of the number of users. This approximation becomes more accurate as the number of users becomes large. The limiting model is therefore "scalable" as it does not suffer from the state space explosion, nor does it require any ad-hoc assumptions.

(ii) Theorem 1 also shows that the dependency between each TCP flow becomes negligible under a large number of flows, i.e., "RED breaks the global synchronization when the number of flows is large."

(iii) The bottleneck capacity being NC, in non-trivial situations, the average (steady-state) throughput of each flow is seen to be approximately C for large N, so that TCP traffic does not realize the benefits commonly associated with statistical multiplexing. Indeed, for N open-loop independent traffic sources each operating at average rate C, the bandwidth required to serve the aggregate flow is typically less than NC under statistical multiplexing. TCP flows, on the other hand, are correlated due to their coupling via the bottleneck router, and multiplexing TCP flows is not as effective.

(iv) The queue length in steady-state can be easily calculated, while the steady-state distribution of the window size and the average window size can be evaluated from well-known TCP models.

(v) For a more accurate description of the queue dynamics, a Central Limit Theorem-type analysis (summarized in Theorem 2) yields the existence of a rv $L_0(t)$ such that

$$Q^{(N)}(t) \simeq Nq(t) + \sqrt{N} L_0(t). \tag{1}$$

(vi) This CLT analysis also reveals that the magnitude of queue fluctuations is proportional to the derivative of the marking probability function. This advocates the use of a smooth marking probability function in line with the RED "gentle" option recently suggested, but in contrast to the original recommendation in [8], [17], [18]. This finding coincides with the oscillatory behavior observed with RED when the average packet drop rate exceeds *max_p*, in the absence of RED's "gentle" modification [19].

The analysis in this paper provides a solid foundation for several assumptions that are made in other models of TCP and for many empirical observations. Some of the contributions have been derived separately in other works, e.g., (iii) in [20] and (vi) in [10] and [21]. However, the fact that these conclusions can all be developed within a *single* model as done here rebukes the arguments that these findings are the byproducts of ad-hoc assumptions or models that are tailored to derive such results.

III. The model

A. A brief review of TCP + ECN/RED dynamics

The TCP congestion control algorithm dynamically adjusts the size of the congestion window (the amount of unacknowledged packets in the network per round-trip) by the following mechanism (assuming ECN/RED is utilized at the bottleneck node). In a round-trip, if all the packets transmitted are not marked, then the size of the congestion window is increased by one packet for the next round-trip. On the other hand, if at least one packet is marked in the round-trip, the congestion window is halved. The probability that the router will mark packets in the buffer depends on the average queue length of the time of packet arrival. The average queue length is calculated by an exponential average filter with large memory to prevent RED from reacting too fast. As a result, consecutive incoming packets into RED are marked with almost identical probability. With this in mind, we now construct a model whose dynamics are similar in spirit to the dynamics of TCP + ECN/RED.

B. The discrete-time model

Time is assumed discrete and slotted in contiguous timeslots of duration equal to the round-trip delay of TCP connections. We consider N traffic sources, all transmitting through a bottleneck RED gateway with ECN enabled in both TCP and RED. The capacity of this bottleneck link is NC packets/slot for some positive constant C. The RED buffer is modeled as an infinite queue, so that no packet losses occur due to buffer overflow, and congestion control is achieved solely through the random marking algorithm in the RED gateway.

Fix $N = 1, 2, \ldots$. We write $X^{(N)}$ to indicate the explicit dependence of the quantity X on the number N of connections.

C. Dynamics

Fix $N = 1, 2, \ldots$, and suppose that each of the N sources (i.e., TCP connections) has an infinite amount of data to transmit and that in each timeslot it transmits as much as allowed by its congestion window in that timeslot. So, for $i = 1, \ldots, N$, let $W_i^{(N)}(t)$ be an integer-valued rv that encodes the number of packets generated by source i (and hence its congestion window) at the beginning of timeslot $[t, t+1)$. We assume the integer $W_i^{(N)}(t)$ to be in the range $\{1, \ldots, W_{\max}\}$ for some finite integer $W_{\max}$. Throughout we assume $W_{\max} \geq 2$ to avoid boundary cases of limited interest.

Upon arrival at the RED gateway, each packet from source i may be marked according to a random marking algorithm (to be specified shortly). We represent this possibility by the $\{0,1\}$-valued rv $M_{i,j}^{(N)}(t+1)$ ($j = 1, \ldots, W_i^{(N)}(t)$) with the interpretation that $M_{i,j}^{(N)}(t+1) = 0$ (resp. $M_{i,j}^{(N)}(t+1) = 1$) if the jth packet from source i is marked (resp. not marked) in the RED buffer. Given that N sources are active, the total number of packets which are accepted into the RED buffer at the beginning of timeslot $[t, t+1)$ is given by $\sum_{i=1}^{N} W_i^{(N)}(t)$.

If $Q^{(N)}(t)$ denotes the number of packets in the buffer at the beginning of timeslot $[t, t+1)$, then $Q^{(N)}(t) + \sum_{i=1}^{N} W_i^{(N)}(t)$ packets are available for transmission in that timeslot. Since the outgoing link operates at the rate of NC packets/timeslot, $\left[Q^{(N)}(t) + \sum_{i=1}^{N} W_i^{(N)}(t) - NC\right]^+$ packets will not be transmitted during timeslot $[t, t+1)$, and remain in the buffer, their transmission being deferred to subsequent timeslots. The number $Q^{(N)}(t+1)$ of packets in the buffer at the beginning of timeslot $[t+1, t+2)$ is therefore given by

$$Q^{(N)}(t+1) = \left[Q^{(N)}(t) - NC + \sum_{i=1}^{N} W_i^{(N)}(t)\right]^+ . \quad (2)$$

D. Statistical assumptions

In order to fully specify the model, we need to specify the *joint* statistics of the rvs $\{M_{i,j}^{(N)}(t+1), W_i^{(N)}(t),\ i = 1, \ldots, N;\ j = 1, 2, \ldots;\ t = 0, 1, \ldots\}$. To do so we introduce the collection of i.i.d. $[0,1]$-*uniform* rvs $\{V_i(t+1),\ V_{i,j}(t+1),\ i, j = 1, \ldots;\ t = 0, 1, \ldots\}$ which are assumed independent of the rvs $W_1^{(N)}(0), \ldots, W_N^{(N)}(0)$ and $Q^{(N)}(0)$. We also introduce a mapping $f^{(N)} : \mathbb{R}_+ \to [0,1]$ which acts as the *marking probability* function of the RED gateway.

The process by which packets are marked is described first: For each $i = 1, \ldots, N$, we define the marking rvs

$$\begin{aligned} & M_{i,j}^{(N)}(t+1) \\ = \ & \mathbf{1}\left[V_{i,j}(t+1) > f^{(N)}(Q^{(N)}(t))\right], \quad j = 1, 2, \ldots \end{aligned} \quad (3)$$

so that the rv $M_{i,j}^{(N)}(t+1)$ is the indicator function of the event that the jth packet from source i is *not* marked in timeslot $[t, t+1)$. Thus, in a round-trip, each packet coming into the router is marked with identical (conditional) probability which depends only on the queue length at the beginning of the timeslot. This model approximates the case where the memory of the queue averaging mechanism is long, which is the case for the recommended parameter settings of RED [18].

Next we introduce the rvs

$$M_i^{(N)}(t+1) = \prod_{j=1}^{W_i^{(N)}(t)} M_{i,j}^{(N)}(t+1) \quad (4)$$

so that $M_i^{(N)}(t+1) = 1$ (resp. $M_i^{(N)}(t+1) = 0$) corresponds to the event that no packet (resp. at least one packet) from source i has been marked in timeslot $[t, t+1)$. The evolution of the window mechanism for source i can now be described through the recursion

$$
\begin{aligned}
& W_i^{(N)}(t+1) \qquad (5)\\
= \; & \min\left(W_i^{(N)}(t)+1, W_{\max}\right) M_i^{(N)}(t+1) \\
& + \min\left(\lceil \frac{W_i^{(N)}(t)}{2} \rceil, W_{\max}\right)(1 - M_i^{(N)}(t+1)).
\end{aligned}
$$

This equation emulates the interaction between TCP and RED as follows: If no packet from source i is marked in timeslot $[t, t+1)$, then the congestion window size in the next timeslot is increased by 1. On the other hand, if one or more packets are marked in timeslot $[t, t+1)$, then the congestion window in the next timeslot is reduced by half. The size of the congestion window is limited by the maximum window size $W_{\max}$ [1].

IV. The asymptotics

The first result of the paper consists in the asymptotics for the normalized buffer content as the number of TCP flows becomes large. This result, contained in Theorem 1, is discussed under the following assumptions (A1)-(A2):

(A1) There exists a continuous function $f : \mathbb{R}_+ \to [0,1]$ such that for each $N = 1, 2, \ldots$,

$$
f^{(N)}(x) = f(N^{-1}x), \quad x \geq 0;
$$

(A2) For each $N = 1, 2, \ldots$, the dynamics (2) and (5) start with the conditions

$$
Q^{(N)}(0) = 0 \quad and \quad W_i^{(N)}(0) = W, \quad i = 1, \ldots, N
$$

for some integer W in the range $\{1, \ldots, W_{\max}\}$.

Assumption (A1) is a structural condition while (A2) is made essentially for technical convenience as it implies that for each $N = 1, 2, \ldots$ and all $t = 0, 1, \ldots$, the rvs $W_1^{(N)}(t), \ldots, W_N^{(N)}(t)$ are *exchangeable*. Assumption (A2) can be omitted but at the expense of a more cumbersome discussion.

Theorem 1: Assume (A1)-(A2) to hold. Then, for each $t = 0, 1, \ldots$, *there exist a (non-random) constant* $q(t)$ *and an* $\{1, \ldots, W_{\max}\}$*-valued rv* $W(t)$ *such that the following holds:*

(i) *The convergence results*

$$
\frac{Q^{(N)}(t)}{N} \xrightarrow{P}_N q(t) \quad \text{and} \quad W_1^{(N)}(t) \Rightarrow_N W(t) \qquad (6)
$$

take place;

(ii) *For any function* $g : \mathbb{N} \to \mathbb{R}$,

$$
\tfrac{1}{N}\sum_{i=1}^N g(W_i^{(N)}(t)) \xrightarrow{P}_N \mathbf{E}\left[g(W(t))\right]. \qquad (7)
$$

(iii) *For any integer* $I = 1, 2, \ldots$, *the rvs* $\{W_i^{(N)}(t),\ i = 1, \ldots, I\}$ *become asymptotically independent as* N *becomes large, with*

$$
\begin{aligned}
& \lim_{N\to\infty} \mathbf{P}\left[W_i^{(N)}(t) = k_i,\ i = 1, \ldots, I\right] \\
= \; & \prod_{i=1}^{I} \mathbf{P}\left[W(t) = k_i\right] \qquad (8)
\end{aligned}
$$

for any $k_1, \ldots, k_I$ *in* $\mathbb{N}$.

Moreover, with initial conditions $q(0) = 0$ *and* $W(0) = W$, *it holds that*

$$
q(t+1) = \left(q(t) - C + \mathbf{E}\left[W(t)\right]\right)^+ \qquad (9)
$$

and

$$
\begin{aligned}
& W(t+1) \qquad (10)\\
=_{st} \; & \min\left(W(t)+1, W_{\max}\right) M(t+1) \\
& + \min\left(\lceil \frac{W(t)}{2} \rceil, W_{\max}\right)(1 - M(t+1)),
\end{aligned}
$$

where

$$
M(t+1) = \mathbf{1}\left[V(t+1) \leq (1 - f(q(t)))^{W(t)}\right] \qquad (11)
$$

for i.i.d. $[0,1]$*-uniform rvs* $\{V(t+1),\ t = 0, 1, \ldots\}$.

A proof of Theorem 1 is available in Appendix A. As should be clear from the discussion given there, Theorem 1 readily flows from a weak Law of Large Numbers for the triangular array

$$
\{W_i^{(N)}(t),\ i = 1, \ldots, N; N = 1, 2, \ldots\}. \qquad (12)
$$

The numerical calculations for the limiting model are very simple. The number of states required for the calculation for each time step is only $W_{\max} + 1$ regardless of N. We can determine $q(t)$ through the following steps:

(i) For $t = 0$, start with some given values $q(0) = 0$ and $W(0) = W$, i.e., $\mathbf{P}\left[W(0) = j\right] = \delta(j, W)$ for $j = 1, \ldots, W_{\max}$, and use $\mathbf{E}\left[W(0)\right] = W$ to calculate $q(1)$ via (9);
(ii) Given $q(t)$ and $\mathbf{P}\left[W(t) = j\right]$ ($j = 1, \ldots, W_{\max}$) for some $t = 0, 1, \ldots$, use (10)-(11) with $q(t)$ to calculate the transition probabilities and $\mathbf{P}\left[W(t+1) = j\right]$ for $j = 1, \ldots, W_{\max}$. Then calculate $\mathbf{E}\left[W(t+1)\right]$;
(iii) Use $\mathbf{E}\left[W(t+1)\right]$ in (ii) to update $q(t+2)$ from (9);
(iv) Increase t by one and repeat Step (ii)-(iv).

V. Steady-state Regime

We now turn our attention to the steady state regime of the limiting two-dimensional recursion (9)-(11), more specifically to the calculation of the limiting queue and average window size in statistical equilibrium, i.e., large t asymptotics. We show that there is a close relationship between this limiting behavior in steady state and the now standard TCP throughput model with stationary loss developed in [2] [22]:

Throughout, we assume the following assumptions (A3)-(A4), where

(A3) The marking function $f : \mathbb{R} \to [0,1]$ is monotonically increasing with

$$
f(0) = 0 \quad \text{and} \quad \lim_{x\to\infty} f(x) = 1;
$$

(A4) The sequence $\{(q(t), W(t)),\ t = 0, 1, \ldots\}$ admits a steady state in the sense that

$$
(q(t), W(t)) \Rightarrow_t (q^\star, W^\star)
$$

[1] If $W_i^{(N)}(0)$ lies in the range $\{1, \ldots, W_{\max}\}$ for each $i = 1, \ldots, N$, then so does $W_i^{(N)}(t)$ for each $t = 0, 1, \ldots$ and the minimum with $W_{\max}$ in the second term of (5) can be omitted.

for some pair $(q^\star, W^\star)$ where $q^\star$ is non-random and $W^\star$ is an $\{1, \ldots, W_{\max}\}$-valued rv.

Although the two-dimensional sequence $\{(q(t), W(t)),\ t = 0, 1, \ldots\}$ is a time-homogeneous Markov chain with values in $\mathbb{R}_+ \times \{1, \ldots, W_{\max}\}$, we do not address here the existence of the limit posited in (A4) as complications arise due the fact that the first component is degenerate (i.e., deterministic). However, under the condition $W_{\max} \leq C$, it is a simple matter to check that (A4) holds with $q^\star = 0$ and $W^\star = W_{\max}$ as would be expected; related details are given in Section V-B (Case 1). Thus, only the case $C < W_{\max}$ needs to be considered.

A. TCP Throughput With Fixed Loss Probability

In order to discuss the limit $(q^\star, W^\star)$ postulated in (A4), we first need to review the results from the TCP throughput model with stationary loss [2] [22]:

Fix γ in $[0, 1]$ and write $\gamma = 1 - p$ with p interpreted as the stationary dropping/marking probability. Consider the recursion

$$\begin{aligned} W_0^\gamma &= W_0; \\ W_{n+1}^\gamma &= \min\left(W_n^\gamma + 1, W_{\max}\right) M_{n+1}^\gamma \\ &\quad + \min\left(\lceil \frac{W_n^\gamma}{2} \rceil, W_{\max}\right)(1 - M_{n+1}^\gamma) \end{aligned}$$

for all $n = 0, 1, \ldots$ and some $\{1, \ldots, W_{\max}\}$-valued rv W_0, with

$$M_{n+1}^\gamma = \mathbf{1}\left[V_{n+1} \leq \gamma^{W_n^\gamma}\right], \quad n = 0, 1, \ldots \tag{13}$$

where the rvs $\{V_{n+1},\ n = 0, 1, \ldots\}$ are i.i.d. $[0, 1]$-uniform rvs which are independent of W_0.

The rvs $\{W_n^\gamma,\ n = 0, 1, \ldots\}$ form a time-homogeneous Markov chain on the finite set $\{1, \ldots, W_{\max}\}$. For γ in the open interval $(0, 1)$, this chain is irreducible, positive recurrent and aperiodic, thus ergodic. Consequently,

$$W_n^\gamma \Rightarrow_n W^\gamma \tag{14}$$

for some $\{1, \ldots, W_{\max}\}$-valued rv W^γ. This rv W^γ satisfies the distributional equation

$$\begin{aligned} W^\gamma &=_{st} \min\left(W^\gamma + 1, , W_{\max}\right) M^\gamma \qquad (15) \\ &\quad + \min\left(\lceil \frac{W^\gamma}{2} \rceil, W_{\max}\right)(1 - M^\gamma) \end{aligned}$$

where

$$M^\gamma = \mathbf{1}\left[V \leq \gamma^{W^\gamma}\right] \tag{16}$$

for some $[0, 1]$-uniform rv V which is independent of the rv W^γ. In fact, the ergodicity of the Markov chain guarantees that the equation (15)-(16) has a solution and that this solution is unique.

For sake of completeness, we also consider the boundary cases: For $\gamma = 0$ (resp. $\gamma = 1$), it is easy to see that (14) also takes place with $W^\gamma = 1$ (resp. $W^\gamma = W_{\max}$). The converse is also true as we now demonstrate: If $W^\gamma = 1$ under (14), then (15)-(16) read

$$1 =_{st} \min\left(1 + M^\gamma, W_{\max}\right)$$

with $M^\gamma = \mathbf{1}[V \leq \gamma]$, so that necessarily $M^\gamma = 0$ under (A4), whence $\gamma = 0$.

On the other hand, if $W^\gamma = W_{\max}$ under (14), then (15)-(16) now reduce to

$$W_{\max} =_{st} W_{\max} M^\gamma + \lceil \frac{W_{\max}}{2} \rceil (1 - M^\gamma)$$

with

$$M^\gamma = \mathbf{1}\left[V \leq \gamma^{W_{\max}}\right].$$

Consequently,

$$\lfloor \frac{W_{\max}}{2} \rfloor =_{st} \lfloor \frac{W_{\max}}{2} \rfloor M^\gamma.$$

so that $M^\gamma = 1$, whence $\gamma = 1$.

Although we can numerically compute the steady-state distribution determined by (15)-(16) (which is a special case of [22]), we take notice that this model is actually that of a single TCP connection with a constant loss probability $1 - \gamma$. This is a well-studied problem (e.g., [2], [3]) with known results. If we replace $\lfloor \frac{W_n^\gamma}{2} \rfloor$ in (15) with $\frac{W_n^\gamma}{2}$, then we can invoke Eqn. (33) in [2] to get the approximation

$$\mathbf{E}\left[W^\gamma\right] \simeq \min\left(W_{\max}, \sqrt{\frac{3}{2(1 - \gamma)}}\right). \tag{17}$$

B. Steady-state regime for the model in Section III

Under Assumption (A4), it is a simple matter to see that

$$(q(t), W(t), M(t + 1)) \Rightarrow_t (q^\star, W^\star, M^\star)$$

with

$$M^\star = \mathbf{1}\left[V \leq (1 - f(q^\star))^{W^\star}\right] \tag{18}$$

where the $[0, 1]$-uniform rv V is independent of the pair $(q^\star, W^\star)$.

Upon letting t go to infinity in (9) and (10), we obtain the relations

$$q^\star = \left(q^\star - C + \mathbf{E}\left[W^\star\right]\right)^+ \tag{19}$$

and

$$\begin{aligned} W^\star &=_{st} \min\left(W^\star + 1, W_{\max}\right) M^\star \\ &\quad + \min\left(\lceil \frac{W^\star}{2} \rceil, W_{\max}\right)(1 - M^\star). \qquad (20) \end{aligned}$$

With $q^\star$ given, the solution to (20) with (18) exists and is unique; it is in fact given by

$$W^\star = W^\gamma \quad \text{with} \quad \gamma = 1 - f(q^\star)$$

where the rv W^γ is defined through (14). Several cases are possible when considering (19) and (20).

Case 1 – $f(q^\star) = 0$: Then, $q^\star$ is finite under (A3), $M^\star = 1$ and (20) reduces to

$$W^\star =_{st} \min\left(W^\star + 1, W_{\max}\right)$$

with unique solution $W^\star = W_{\max}$ (in agreement with the discussion in Section V-A). In that case, (19) gives

$$q^\star = \left(q^\star - C + W_{\max}\right)^+. \tag{21}$$

Therefore, either $q^\star = 0$ in which case $W_{\max} \le C$ as should be expected, or $q^\star > 0$ (still with $f(q^\star) = 0$), in which case $C = W_{\max}$. However, under the condition $W_{\max} \le C$, it is clear that if $Q^{(N)}(0) = Q \ge 0$ for all $N = 1, 2, \ldots$, then $Q^{(N)}(t) \le Q$ and the conclusion $q(t) = 0$ holds for all $t = 0, 1, \ldots$, so that $q^\star = 0$. In other words, if $q^\star$ in (A4) is such that $f(q^\star) = 0$, then necessarily $q^\star = 0$. ■

Case 2 – $f(q^\star) = 1$: Then $q^\star > 0$, $M^\star = 0$ and (20) now reduces to

$$W^\star =_{st} \min\left(\lceil \frac{W^\star}{2} \rceil, W_{\max}\right)$$

with only solution $W^\star = 1$ (also in agreement with the discussion in Section V-A). ■

Case 3 – $0 < f(q^\star) < 1$: Then, $0 < q^\star < \infty$ and from (19) it is necessarily the case that $\mathbf{E}[W^\star] = C$. Thus, the existence of a steady state requires at the very least that the equation[2]

$$\mathbf{E}[W^\gamma] = C, \quad \gamma \in [0, 1] \tag{22}$$

has a unique solution, say $\gamma^\star$, in which case we must have

$$\gamma^\star = 1 - f(q^\star).$$

By known results on finite state Markov chains [23], the mapping $\gamma \to \mathbf{E}[W^\gamma]$ is continuous on $[0, 1]$ with

$$\mathbf{E}[W^\gamma]_{\gamma=0} = 1 \quad \text{and} \quad \mathbf{E}[W^\gamma]_{\gamma=1} = W_{\max}.$$

By continuity, $\{\mathbf{E}[W^\gamma],\ \gamma \in [0, 1]\}$ must contain the interval $[1, W_{\max}]$. On the other, it is always the case that

$$1 \le \mathbf{E}[W^\gamma] \le W_{\max}, \quad \gamma \in [0, 1],$$

and we conclude that $\{\mathbf{E}[W^\gamma],\ \gamma \in [0, 1]\} = [1, W_{\max}]$. Therefore, there exists at least one solution to (22) *provided* $1 \le C \le W_{\max}$. The uniqueness of the solution would be ensured by the strict monotonicity of the mapping $\gamma \to \mathbf{E}[W^\gamma]$ illustrated in Figure 1 (and also suggested by the approximation (17)). ■

VI. A CENTRAL LIMIT THEOREM

In this section, we present a Central Limit Theorem (CLT) which complements the limiting results obtained earlier. The discussion is carried out in the setup of Section IV, but with Assumption (A1) strengthened to read as Assumption (A1b), where

(A1b) Assumption (A1) holds with mapping $f : \mathbb{R}_+ \to [0, 1]$ which is continuously differentiable, i.e., its derivative $f' : \mathbb{R}_+ \to \mathbb{R}$ exists and is continuous.

For each $t = 0, 1, \ldots$, let $q(t)$ and $W(t)$ be as in Theorem 1, and set

$$L_0^{(N)}(t) := \frac{Q^{(N)}(t)}{N} - q(t)$$

[2]Equation 22 forms the basis for item (iii) in Section II.

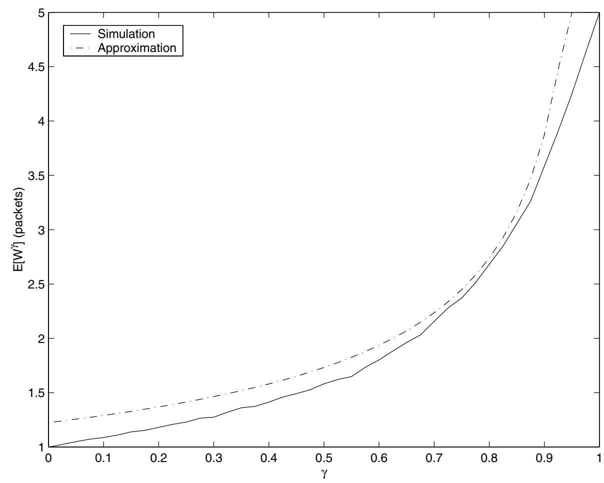

Fig. 1. The mapping $\gamma \to \mathbf{E}[W^\gamma]$ when $W_{\max} = 5$ vs. the approximation (17).

and

$$\bar{L}^{(N)}(t) = \frac{1}{N}\left(\sum_{i=1}^{N} W_i^{(N)}(t) - \mathbf{E}[W(t)]\right).$$

Theorem 2: Assume (A1b)-(A2) to hold. Then, for each $t = 0, 1, \ldots$, *there exists an* $\mathbb{R}^2$*-valued rv* $\mathbf{L}(t) = (L_0(t), \bar{L}(t))$ *such that the convergence*

$$\sqrt{N}\left(L_0^{(N)}(t), \bar{L}^{(N)}(t)\right) \Rightarrow_N \mathbf{L}(t) \tag{23}$$

holds. Moreover, the distributional recurrence

$$L_0(t+1) =_{st} \begin{cases} 0 & K(t) > 0 \\ L_0(t) + \bar{L}(t) & K(t) < 0 \\ \left(L_0(t) + \bar{L}(t)\right)^+ & K(t) = 0 \end{cases} \tag{24}$$

holds where we have set

$$K(t) = C - q(t) - \mathbf{E}[W(t)].$$

For any $t = 1, 2, \ldots$, *if there exists an integer* $r < t$ *such that* $K(r) > 0$ *and* $K(s) \ne 0$ *for all* $s = r, r+1, \ldots, t$, *then the rv* $L_0(t)$ *is Gaussian.*

The convergence (23) suggests the approximation (1). We can interpret $K(t)$ as the residual capacity per user in the limit in timeslot $[t, t+1)$. If there exists extra capacity for the average user rate to increase ($K(t) > 0$), then there is no fluctuation in the limiting queue. On the other hand, when there is congestion ($K(t) < 0$), the (non-trivial) limiting distribution can be found recursively. Some technical difficulties arise in the special case $K(t) = 0$.

A distributional recursion is available for $\bar{L}(t)$, but is much more complicated to describe due to feedback interaction present in the system. Due to space constraints, this distributional recursion and the proof of Theorem 2 will be omitted; details can be found in [24]. However, we do note the following interesting fact: As part of the proof of Theorem 2 we find that

$$\bar{L}(t+1) =_{st} c(t) f'(q(t)) L_0(t) + \xi(t)$$

for some rv $\xi(t)$ and non-random constant $c(t)$. It turns out that we have the decomposition $\xi(t) = \eta(t) + \zeta(t)$ with $\zeta(t)$ a zero mean Gaussian rv which is independent of the pair $(L_0(t), \eta(t))$ and $\eta(t)$ is associated with a CLT-type result for the array (12). Thus, in general neither component of $\mathbf{L}(t)$ may be normally distributed. Moreover, the magnitude of the queue fluctuations is seen to be proportional to the derivative of f around the limiting (normalized) queue size $q(t)$. Since the original drop probability function of RED is discontinuous around *max_thresh*, our analysis is very much in line with the reported oscillatory queue behavior when the average queue exceeds *max_thresh* and with the finding that the addition of the "gentle" option can improve the queue behavior [19].

To understand why the derivative of the marking function plays such an important role, note that $f(\frac{Q^{(N)}(t)}{N})$ is the only feedback information from the RED gateway to the TCP sources. However, this feedback information also fluctuates around its limiting value $f(q(t))$. It is easy to imagine that the uncertainty in the feedback information will also lead to fluctuations in the limiting queue as is suggested by the following result (also known as the *Delta Method* [26, p. 214]).

Lemma 1: If $f : \mathbb{R}_+ \to [0,1]$ is differentiable with derivative continuous at $x = q(t)$, then the convergence $\sqrt{N}(\frac{Q^{(N)}(t)}{N} - q(t)) \Rightarrow_N L_0(t)$ implies $\sqrt{N}(f(\frac{Q^{(N)}(t)}{N}) - f(q(t))) \Rightarrow_N f'(q(t))L_0(t)$.

VII. Simulation Results

In this section, we present results from (Monte-Carlo) simulations of the model presented in Section III *and* from NS-2 simulations [6] to illustrate the behaviors suggested by both Theorems 1 and 2. For the NS-2 simulations, we use the system shown in Figure 2. Each server establishes a TCP Reno connection to a corresponding client, thereby competing for the capacity in the ECN/RED gateway. Each TCP has a fixed packet size of 1500 bytes and a maximum window size of 200 packets. The marking probability function in the ECN/RED gateway is specified as in (A1) with $f : \mathbb{R}_+ \to [0,1]$ taken to be

$$f(x) = \min\left(0.01(x-1)^+, 1\right), \quad x \geq 0. \tag{25}$$

We choose this simple marking probability function with only one major slope in order to later demonstrate the relationship between the magnitude of the queue fluctuations and the slope of f as mentioned in the end of Section VI.

The "time constant" parameter w_q for the Exponential Weighted Moving Average is set to 0.002, similar to the recommended value in [18]. Every round equals the round-trip propagation delay of 200 milliseconds. At the beginning of each round, we collect the instantaneous queue length in the ECN/RED buffer for a total duration of 200 seconds. Figure 3 shows the queue length normalized by the number of connection (N) as a function of time. We note a behavior similar to that discussed in Theorem 1 as fluctuations in the normalized queue length decrease with an increasing number of connections. Figure 3 also suggests the existence of a steady-state for the limiting model, with a steady-state normalized queue length being constant at approximately 4.85 packets/user, corresponding to the steady-state marking probability of $0.0385 = f(4.85)$.

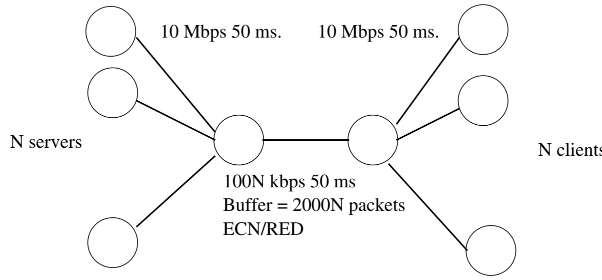

Fig. 2. The simulation setup in NS-2.

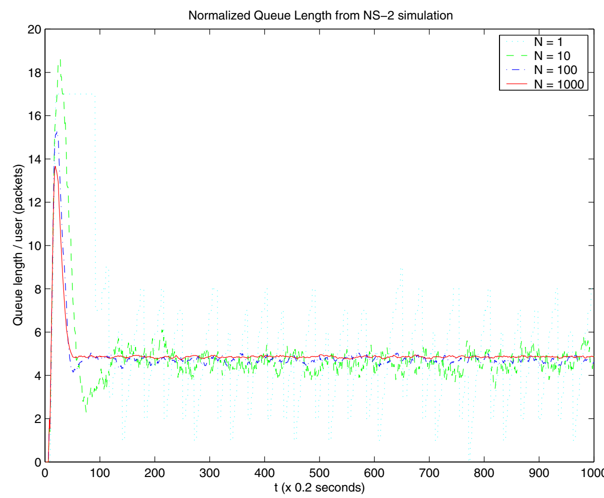

Fig. 3. The normalized queue length of the ECN/RED gateway in NS-2 simulation.

To simulate the model described in Section III, we use the same parameter setup as in the NS simulation, i.e., $W_{\max} = 200$, simulation time of 1000 timeslots and the same marking function. The capacity per user (C) of the bottleneck router can be calculated from (17) and (22) when the marking probability p is 0.0385 (obtained from the NS simulation, so that $\gamma = 1 - p = .9615$). A simple calculation yields $C = 6.24$ packets/timeslot. The simulation results are shown in Figure 4. A quick comparison to Figure 3 indicates a qualitative similarity in that the fluctuations decrease as the number of users increases. Further inspection reveals that the average normalized queue length is around 4.93 packets/user, very close to 4.85 packets/user produced by the NS simulation. Therefore, the limiting stochastic model appears to capture the essential behavior of queue dynamics in ECN/RED gateways, although the model exhibits somewhat greater fluctuations than in the NS simulation. This is due to the fact that all flows in the model are synchronized at the beginning of each timeslot, i.e., they adapt at the same time while in the NS simulator, the flows react asynchronously to the mark from the RED gateway.

To gauge the rate of convergence, we assume that the queue

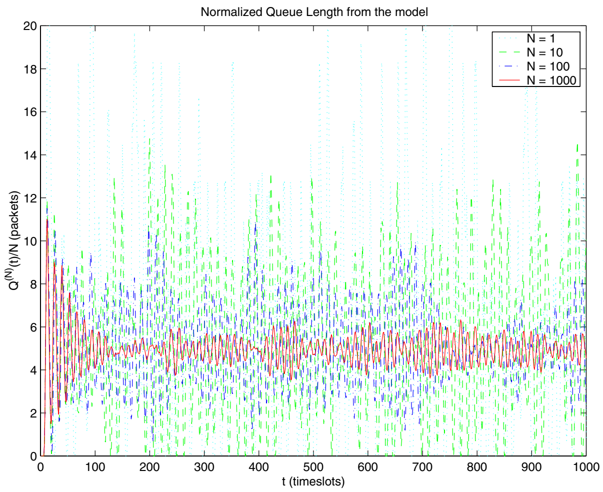

Fig. 4. The normalized queue length of the model.

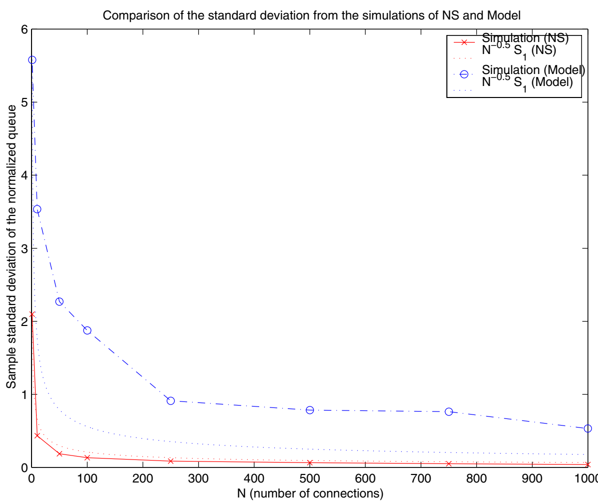

Fig. 5. Sample standard deviation of the normalized queue.

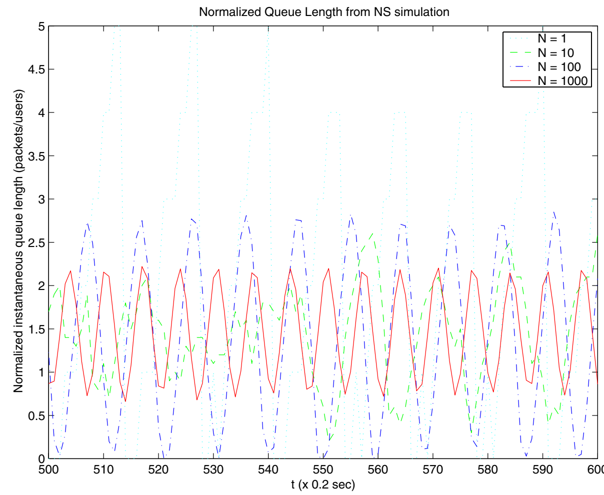

Fig. 6. The normalized queue length of the ECN/RED gateway in NS-2 simulation with the marking probability function (26).

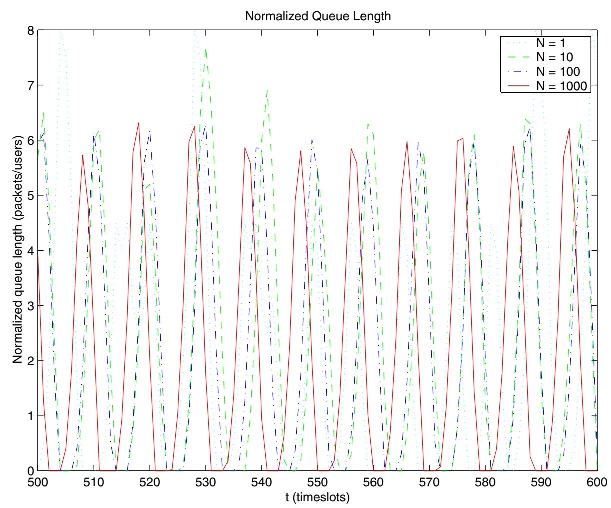

Fig. 7. The normalized queue length of the model with the marking probability function (26).

is in steady state after the first 100 samples and that the magnitude of the queue fluctuations around its steady state mean is Gaussian[3] and ergodic. Therefore, the steady-state standard deviation of the queue can be approximated from the sample standard deviation of the queue at timeslot 101 and beyond. The comparison between the sample standard deviation from the model and NS simulations is displayed in Figure 5. They clearly follow a similar trend. We also expect from Theorem 2 that the standard deviation will decrease as $1/\sqrt{N}$ for large N. Let S_N denote the sample standard deviation of the normalized queue when the number of users is N. We can see from Figure 5 that $S_1/\sqrt{N}$ provides a good approximation of the standard deviation S_N for large N.

To demonstrate the relationship between the slope of the marking probability function and the magnitude of the queue fluctuation (as mentioned at the end of Section VI), we use the same network setup as before but with a slope increased ten-fold, i.e.,

$$f(x) = \min\left(0.1(x-1)^+, 1\right), \quad x \geq 0. \tag{26}$$

Figures 6 and 7 show the simulation results in the case of the steeper function (26). In both the Monte-Carlo and NS-2 simulations it is clear that the magnitude of queue fluctuation is much larger than in the original simulation. As the number of flows increases, the convergence becomes much slower than in the original setup. In the control-theoretic view of [10], this phenomenon can be interpreted as the control system becoming oscillatory with too large a feedback gain.

VIII. Application to Network Dimensioning

We now consider a simple application of these limiting results: An ISP currently services up to N TCP flows at peak hour through an ECN/RED access gateway connecting to the

[3] By Theorem 2 the fluctuation $L_0(t)$ is Gaussian if $K(s) \neq 0$, $s = 0, \ldots, t$.

core network with link speed of NC packets/sec. The network manager can roughly determine the buffer utilization in the ECN/RED gateway as follows:

(i) Determine the marking probability per flow ($p = f(q)$) from the relation $C = \mathbf{E}\left[W^{1-p}\right]$ by using a TCP throughput formula such as (17). If $W_{\max} \leq C$, the steady-state analysis in Section V suggests that the link is already over-provisioned and no congestion control action is required, i.e., the marking probability per flow is simply $f(q) = 0$.

(ii) Calculate the limiting queue length q in steady state by solving $p = f(q)$;

(iii) Approximate the queue length distribution in steady state via the CLT complement. If the steady state exists, the CLT complement determines the distribution of the queue size fluctuations around q. The delay and overflow distributions can also be approximated via the CLT complement.

Although these limiting results apply only to TCP flows with identical round-trip times, they would be of use in a number situations, e.g., an intercontinental Internet link where this link is typically a bottleneck, its large propagation delay dominates the round-trip and the number of flows is extremely large.

IX. Conclusions

We have developed a stochastic model of an ECN/RED gateway under competing TCP flows. We have shown that, as the number of flows grows large, the aggregate behavior of the queue can be described by a two-dimensional recursion and TCP flows become asymptotically independent, thus no longer synchronized. In addition, a method to calculate the buffer utilization at the steady-state is proposed. The CLT complement yields further insight on the relationship between the queue fluctuations and the marking probability function.

Although we have yet to prove the existence of a steady state regime for the limiting recursion identified here, the limited simulation results are compatible with the existence of such a steady state under some conditions on the marking probability function.

Future work on this class of models includes (i) a proof of the existence of a steady state for the limiting dynamics and its evaluation; (ii) the incorporation of additional features, e.g., the slow-start phase, random round-trip delays and heterogeneous populations of TCP flows; and (iii) the modeling of non-responsive UDP flows and short-lived TCP flows. Some progress on the issues in (ii) is reported in [25].

References

[1] V. Jacobson, "Congestion avoidance and control," in *Proceedings of SIGCOMM'88 Symposium*, Aug. 1988, pp. 314–332.

[2] J. Padhye, V. Firoiu, D. Towsley, and J. Kurose, "Modeling TCP Reno performance: A simple model and its empirical validation," *IEEE/ACM Transactions on Networking*, Apr. 2000.

[3] M. Mathis, J. Semske, J. Mahdavi, and T. Ott, "The macroscopic behavior of TCP congestion avoidance algorithm," *Computer Communication Review*, vol. 27, no. 3, July 1997.

[4] E. Altman, K. Avrachenkov, and C. Barakat, "TCP in presence of bursty losses," in *Proceedings of SIGMETRICS*, 2000.

[5] V. Misra, W. Gong, and D. Towsley, "Stochastic differential equation modeling and analysis of TCP windowsize behavior," in *Proceedings of Performance*, 1999.

[6] UCB/VINT/LBNL Network Simulator (ns) version 2.0, `http://www.isi.edu/nsnam`.

[7] L. Zhang and D. Clark, "Oscillating behavior of network traffic: A case study simulation," *Internetworking: Research and Experience*, vol. 1, no. 2, pp. 101–112, 1990.

[8] S. Floyd and V. Jacobson, "Random early detection gateways for congestion avoidance," *IEEE/ACM Transactions on Networking*, vol. 1, no. 4, pp. 397–413, Aug. 1995.

[9] S. Floyd, "TCP and explicit congestion notification," *Computer Communication Review*, vol. 24, no. 5, pp. 10–23, Oct. 1994.

[10] C. Hollot, V. Misra, D. Towsley, and W. Gong, "A control theoretic analysis of RED," in *Proceedings of IEEE INFOCOM*, 2001.

[11] A. Abouzeid and S. Roy, "Analytic understanding of RED gateways with multiple competing TCP flows," in *Proceedings of IEEE GLOBECOM*, 2000.

[12] H. M. Alazemi, A. Mokhtar, and M. Azizoglu, "Stochastic modeling of random early detection gateways in TCP networks," in *Proceedings of IEEE GLOBECOM*, 2000.

[13] D. Hong and D. Lebedev, "Many TCP user asymptotic analysis of the AIMD model," Tech. Rep. RR-4229, INRIA, July 2001.

[14] P. Tinnakornsrisuphap and A. M. Makowski, "Queue dynamics of RED gateways under large number of TCP flows," in *Proceedings of IEEE GLOBECOM*, 2001.

[15] F. Baccelli, D. R. McDonald and J. Reynier, "A mean-field model for multiple TCP connections through a buffer implementing RED," Tech. Rep. RR-4449, INRIA, April 2002.

[16] P. Tinnakornsrisuphap and A. M. Makowski, "TCP Modeling via Limit Theorems," Tech. Rep., Institute for Systems Research, University of Maryland, 2002.

[17] S. Floyd and K. Fall, "Router mechanisms to support end-to-end congestion control," Tech. Rep., LBNL, 1997.

[18] S. Floyd, "RED: Discussions of setting parameters," `http://www.icir.org/floyd/REDparameters.txt`.

[19] V. Firoiu and M. Borden, "A study of active queue management for congestion control," in *Proceedings of IEEE INFOCOM*, 2000.

[20] T. Bu and D. Towsley, "Fixed Point Approximation for TCP behavior in an AQM Network," in *Proceedings of ACM SIGMETRICS*, 2001.

[21] S. H. Low, F. Paganini, J. Wang, S. Adlakha and J. C. Doyle, "Dynamics of TCP/RED and a scalable control," in *Proceedings of IEEE INFOCOM*, 2002.

[22] J. Padhye, V. Firoiu, and D. Towsley, "A stochastic model of TCP Reno congestion avoidance and control," Tech. Rep. 99-02, Department of Computer Science, University of Massachusetts, Amherst, 1999.

[23] D.–J. Ma, A.M. Makowski and A. Shwartz, "Stochastic approximations for finite–state Markov chains, *Stochastic Processes and Their Applications*, vol. 35, pp. 27-45, 1990.

[24] P. Tinnakornsrisuphap and A.M. Makowski, "On the behavior of ECN/RED gateways under a large number of TCP flows: Limit theorems," Tech. Rep., Institute for Systems Research, University of Maryland, 2003.

[25] P. Tinnakornsrisuphap, R. La and A. M. Makowski, "Characterization of general TCP traffic under a large number of flows regime," submitted for inclusion in the program of ICC'03, Anchorage (AL), May 2003.

[26] A.F. Karr, *Probability*, Springer-Verlag, 1993.

A. A Proof of Theorem 1

A. Some useful facts

To facilitate the presentation of the proof of Theorem 1, we begin with a few simple and useful facts. Fix $i = 1, \ldots, N$ and consider an arbitrary mapping $g : \mathbb{N} \to \mathbb{R}$: It follows from (5) that

$$\begin{aligned} & g(W_i^{(N)}(t+1)) \\ = \quad & M_i^{(N)}(t+1) g(\min\left(W_i^{(N)}(t) + 1, W_{\max}\right)) \\ + \quad & (1 - M_i^{(N)}(t+1)) g(\min\left(\lceil \frac{W_i^{(N)}(t)}{2} \rceil, W_{\max}\right)). \end{aligned}$$

Let $\mathcal{F}_t$ denote the σ-field generated by the rvs $\{Q^{(N)}(0), W_i^{(N)}(0), V_i(s), V_{i,j}(s),\ i,j = 1,2,\ldots;\ s = 1,\ldots,t\}$. The rvs $Q^{(N)}(t)$ and $W_i^{(N)}(t)$ ($i = 1,\ldots,N$) being all $\mathcal{F}_t$-measurable, it holds under the enforced independence assumptions that

$$\mathbf{E}\left[M_{i,j}^{(N)}(t+1)|\mathcal{F}_t\right] = 1 - f^{(N)}(Q^{(N)}(t)), \quad j = 1,2,\ldots$$

so that

$$\mathbf{E}\left[M_i^{(N)}(t+1)|\mathcal{F}_t\right] = Z_i^{(N)}(t) \tag{27}$$

by conditional independence, where we have set

$$Z_i^{(N)}(t) = \left(1 - f^{(N)}(Q^{(N)}(t))\right)^{W_i^{(N)}(t)}. \tag{28}$$

It is now plain that

$$M_i^{(N)}(t+1) =_{st} \mathbf{1}\left[V_i(t+1) \leq Z_i^{(N)}(t)\right]. \tag{29}$$

It readily follows from (27) that

$$\mathbf{E}\left[g(W_i^{(N)}(t+1))|\mathcal{F}_t\right] = F_g(Z_i^{(N)}(t), W_i^{(N)}(t)) \tag{30}$$

where the mapping $F_g : [0,1] \times \mathbb{N} \to \mathbb{R}$ is associated with g through

$$\begin{aligned} F_g(z,w) &= zg(\min(w+1, W_{\max})) \\ &\quad +(1-z)g(\min(\lceil \frac{w}{2} \rceil, W_{\max})). \end{aligned} \tag{31}$$

Upon taking expectations on both sides in the relation above, we get

$$\begin{aligned} &\mathbf{E}\left[g(W_i^{(N)}(t+1))\right] \\ = \ &\mathbf{E}\left[F_g(Z_i^{(N)}(t), W_i^{(N)}(t))\right]. \end{aligned} \tag{32}$$

B. A Weak Law of Large Numbers

For each $t = 0,1,\ldots$, the statements **[A:t]**, **[B:t]**, **[C:t]** and **[D:t]** below refer to the following convergence statements:

[A:t] For some non-random $q(t)$, it holds that

$$\frac{Q^{(N)}(t)}{N} \xrightarrow{P}_N q(t); \tag{33}$$

[B:t] For some $\{1,\ldots,W_{\max}\}$-valued rv $W(t)$, it holds that

$$W_1^{(N)}(t) \Rightarrow_N W(t); \tag{34}$$

[C:t] For any integer $I = 1,2,\ldots$, the rvs $\{W_i^{(N)}(t),\ i = 1,\ldots,I\}$ become asymptotically independent with large N as described by (8) where $W(t)$ is the rv occurring in **[B:t]**;

[D:t] For any mapping $g : \mathbb{R} \to \mathbb{R}$, the convergence (7) holds with $W(t)$ the rv occurring in **[B:t]**.

Through a series of lemmas, we prove the validity of the statements **[A:t]**–**[D:t]** for all $t = 0,1,\ldots$. We do so by induction on t and in the process we establish Theorem 1.

Lemma 2: Under (A1), if **[A:t]** *and* **[B:t]** *hold for some* $t = 0,1,\ldots$, *then* **[B:t+1]** *holds with* $W(t+1)$ *related to* $W(t)$ *by (10).*

Proof: Together the convergence **[A:t]** and **[B:t]** imply [26, Thm. 5.28, p. 150] the joint convergence $(N^{-1}Q^{(N)}(t), W_1^{(N)}(t)) \Rightarrow_N (q(t), W(t))$. Next the continuity of the mapping f implies that of $(x,w) \to (1-f(x))^w$ on $\mathbb{R}_+ \times (0,\infty)$, so that

$$(Z_1^{(N)}(t), W_1^{(N)}(t)) \Rightarrow_N (Z(t), W(t)) \tag{35}$$

by the Continuous Mapping Theorem [26, Thm. 5.29, p. 150] with $Z(t) = (1 - f(q(t)))^{W(t)}$.

Consider (32) for an arbitrary mapping $g : \mathbb{N} \to \mathbb{R}$, and observe that the mapping F_g defined by (31) is continuous on $[0,1] \times \mathbb{N}$ [4]. Consequently, the Continuous Mapping Theorem can again be invoked to yield

$$F_g(Z_1^{(N)}(t), W_1^{(N)}(t)) \Rightarrow_N F_g(Z(t), W(t)), \tag{36}$$

whence

$$\lim_{N\to\infty} \mathbf{E}\left[F_g(Z_1^{(N)}(t), W_1^{(N)}(t))\right] = \mathbf{E}\left[F_g(Z(t), W(t))\right]$$

by the Bounded Convergence Theorem [26, Thm. 4.16, p. 108]. Combining this last convergence with (32), we get

$$\lim_{N\to\infty} \mathbf{E}\left[g(W_1^{(N)}(t+1))\right] = \mathbf{E}\left[F_g(Z(t), W(t))\right] \tag{37}$$

and the mapping g being arbitrary, it follows immediately that $W_1^{(N)}(t+1) \Rightarrow_N W(t+1)$ for some $\{1,\ldots,W_{\max}\}$–valued rv $W(t+1)$, with

$$\mathbf{E}\left[g(W(t+1))\right] = \mathbf{E}\left[F_g(Z(t), W(t))\right]. \tag{38}$$

A moment of reflection and a comparison to the analysis in (30)-(32) will convince the reader that (38) is equivalent to (10) and (11). ■

Lemma 3: Under (A1), if **[A:t]** *and* **[D:t]** *hold for some* $t = 0,1,\ldots$, *then* **[A:t+1]** *also holds.*

Proof: Using **[A:t]** and **[D:t]** (with $g(x) = x$), we conclude that

$$\begin{aligned} &\frac{Q^{(N)}(t)}{N} - C + \tfrac{1}{N}\textstyle\sum_{i=1}^N W_i^{(N)}(t) \\ \xrightarrow{P}_N \quad &q(t) - C + \mathbf{E}\left[W(t)\right] \end{aligned} \tag{39}$$

and the desired result is now a simple consequence of the continuity of the function $x \to x^+$ since

$$\frac{Q^{(N)}(t+1)}{N} = \left[\frac{Q^{(N)}(t)}{N} - C + \tfrac{1}{N}\textstyle\sum_{i=1}^N W_i^{(N)}(t)\right]^+$$

for all $N = 1,2,\ldots$. ■

The proof of Lemma 3 also shows that

$$\frac{Q^{(N)}(t+1)}{N} \xrightarrow{P}_N q(t+1)$$

with non-random $q(t+1)$ determined by (9).

Lemma 4: Under (A1)–(A2), if **[A:t]**, **[B:t]** *and* **[C:t]** *hold for some* $t = 0,1,\ldots$, *then* **[C:t+1]** *also holds.*

[4] This continuity is with respect to the product topology on $[0,1] \times \mathbb{N}$ where $\mathbb{N}$ is topologized according to the usual discrete topology.

Proof: Fix a positive integer I. The rvs $V_1(t+1),\ldots,V_I(t+1)$ are i.i.d. $[0,1]$-uniform rvs which are independent of $\mathcal{F}_t$. Thus, upon making use of the representation (5) with (29), we see that the rvs $W_1^{(N)}(t+1),\ldots,W_I^{(N)}(t+1)$ are mutually independent given $\mathcal{F}_t$. Consequently, for arbitrary mappings $g_1,\ldots,g_I : \mathbb{N} \to \mathbb{R}$, we get

$$\begin{aligned} & \mathbf{E}\left[\prod_{i=1}^{I} g_i(W_i^{(N)}(t+1)|\mathcal{F}_t\right] \\ = & \prod_{i=1}^{I} \mathbf{E}\left[g_i(W_i^{(N)}(t+1)|\mathcal{F}_t\right] \\ = & \prod_{i=1}^{I} F_{g_i}(Z_i^{(N)}(t), W_i^{(N)}(t)) \end{aligned}$$

with the help of (30) and (31).

Now it follows from (8) in **[C:t]** that the joint convergence

$$(W_1^{(N)}(t),\ldots,W_I^{(N)}(t)) \Rightarrow_N (W_1(t),\ldots,W_I(t))$$

holds with limiting rvs $W_1(t),\ldots,W_I(t)$ which are i.i.d. rvs, each distributed according to $W(t)$. As in the proof of Lemma 2, the arguments leading to the convergence (36) also lead to

$$\begin{aligned} & (F_{g_1}(Z_1^{(N)}(t), W_1^{(N)}(t)),\ldots,F_{g_I}(Z_I^{(N)}(t), W_I^{(N)}(t))) \\ & \Rightarrow_N (F_{g_1}(Z_1(t), W_1(t)),\ldots,F_{g_I}(Z_I(t), W_I(t))) \end{aligned}$$

with the limiting rvs $(Z_1(t), W_1(t)),\ldots,(Z_I(t), W_I(t))$ being i.i.d. rvs, each distributed according to the pair $(Z(t), W(t))$. Therefore, by the Bounded Convergence Theorem,

$$\begin{aligned} & \lim_{N\to\infty} \mathbf{E}\left[\prod_{i=1}^{I} g_i(W_i^{(N)}(t+1)\right] \\ = & \lim_{N\to\infty} \mathbf{E}\left[\prod_{i=1}^{I} F_{g_i}(Z_i^{(N)}(t), W_i^{(N)}(t))\right] \\ = & \mathbf{E}\left[\prod_{i=1}^{I} F_{g_i}(Z_i(t), W_i(t))\right] \\ = & \prod_{i=1}^{I} \mathbf{E}\left[F_{g_i}(Z_i(t), W_i(t))\right] \\ = & \prod_{i=1}^{I} \mathbf{E}\left[g_i(W_i(t+1)\right] \end{aligned} \quad (40)$$

where the last equality made use of the relation (38). The desired result **[C:t+1]** now follows from (40) given that the mappings $g_1,\ldots,g_I$ are arbitrary. ■

Lemma 5: Under (A1)–(A2), if **[A:t]**, **[B:t]** *and* **[C:t]** *hold for some* $t=0,1,\ldots$, *then* **[D:t]** *holds.*

Proof: Pick a mapping $g : \mathbb{N} \to \mathbb{R}$. We begin with the observation that under (A2) the rvs $W_i^{(N)}(t),\ldots,W_N^{(N)}(t)$ are exchangeable. As a result, we get

$$\begin{aligned} & \mathrm{var}\left[\tfrac{1}{N}\textstyle\sum_{i=1}^{N} g(W_i^{(N)}(t))\right] \\ = & N^{-2}\sum_{i=1}^{N} \mathrm{var}[g(W_i^{(N)}(t))] \\ & + N^{-2}\sum_{i,j=1, i\neq j}^{N} \mathrm{cov}[g(W_i^{(N)}(t)), g(W_j^{(N)}(t))] \\ = & N^{-1}\mathrm{var}[g(W_1^{(N)}(t))] \\ & + \frac{N-1}{N}\mathrm{cov}[g(W_1^{(N)}(t)), g(W_2^{(N)}(t))]. \end{aligned} \quad (41)$$

Now let N go to infinity in (41): The validity of **[C:t]** and the Bounded Convergence Theorem already imply

$$\begin{aligned} & \lim_{N\to\infty} \mathrm{cov}[g(W_1^{(N)}(t)), g(W_2^{(N)}(t))] \\ = & \mathrm{cov}[g(W_1(t)), g(W_2(t))] = 0 \end{aligned}$$

by asymptotic independence. On the other hand, with $G := \max\{|g(x)|,\ x = 1,\ldots,W_{\max}\}$, we have $\sup_N \mathrm{var}[g(W_1^{(N)}(t))] < G^2$. Combining these observations we readily see that

$$\lim_{N\to\infty} \mathrm{var}\left[\tfrac{1}{N}\textstyle\sum_{i=1}^{N} g(W_i^{(N)}(t))\right] = 0,$$

whence, by Chebyshev's Inequality,

$$\tfrac{1}{N}\textstyle\sum_{i=1}^{N} g(W_i^{(N)}(t)) - \mathbf{E}\left[\tfrac{1}{N}\textstyle\sum_{i=1}^{N} g(W_i^{(N)}(t))\right] \xrightarrow{P}{}_N 0.$$

This last convergence is equivalent to

$$\tfrac{1}{N}\textstyle\sum_{i=1}^{N} g(W_i^{(N)}(t)) - \mathbf{E}\left[g(W_1^{(N)}(t))\right] \xrightarrow{P}{}_N 0$$

by exchangeability, and the desired convergence result (7) is now immediate once we remark under **[B:t]** that $\lim_{N\to\infty} \mathbf{E}\left[g(W_1^{(N)}(t))\right] = \mathbf{E}\left[g(W(t)\right]$. ■

To conclude the proof of Theorem 1, we note that under (A1)-(A2) the statements **[A:t]**–**[D:t]** trivially hold for $t=0$ with $q(0)=0$ and $W(0)=W$. Moreover, if **[A:t]**–**[C:t]** hold for some $t=0,1,\ldots$, then so do the statements **[D:t]** **[B:t+1]**, **[A:t+1]** and **[C:t+1]** by Lemma 5, Lemma 2, Lemma 3 and Lemma 4, respectively. Consequently, the statements **[A:t]**–**[D:t]** do hold for all $t=0,1,\ldots$ by induction and the validity of Claims (i)-(iii) of Theorem 1 is now established. The dynamics (9) is a byproduct of the proof of Lemma 3, while (10)-(11) are already contained in Lemma 2.